

## Test Functions and Distributions



Test Functions and Linear Functionals  $\mathcal{D}(\mathbb{R}^n) := C_0^{\infty}(\mathbb{R}^n)$ (Space of Test Functions in  $\mathbb{R}^n$ )

 $\mathcal{D}(\mathbb{R}^n)$  is a (complex) vector space.

Definition. A linear functional on  $\mathcal{D}(\mathbb{R}^n)$  is a map

 $T\colon \mathcal{D}(\mathbb{R}^n)\to\mathbb{C}$ 

such that

$$T(\lambda\varphi_1 + \mu\varphi_2) = \lambda T\varphi_1 + \mu T\varphi_2$$

for  $\varphi_1, \varphi_2 \in \mathcal{D}$ ,  $\lambda, \mu \in \mathbb{C}$ .



## Examples of Linear Functionals

Linear functionals on  $\mathcal{D}(\mathbb{R})$ :

(i) 
$$T\varphi := \int_0^\infty \varphi(x) \, dx$$
  
(ii)  $T\varphi := \varphi(0)$   
(iii)  $T\varphi := \varphi'(1)$ 

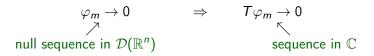
Linear functionals on  $\mathcal{D}(\mathbb{R}^n)$ :

(i) 
$$T\varphi := \int_{\mathbb{R}^n} \varphi(x) dx$$
  
(ii)  $T\varphi := \int_S \varphi d\sigma$ , where S is a surface in  $\mathbb{R}^n$   
(iii)  $T\varphi := \int_S \operatorname{grad} \varphi d\vec{\sigma}$ 



#### Continuous Linear Functionals

Definition. A linear functional T is said to be continuous if



A continuous linear functional on  $\mathcal{D}(\mathbb{R}^n)$  is called a distribution.

The set of all distributions is denoted by  $\mathcal{D}'(\mathbb{R}^n)$ .  $\mathcal{D}'(\mathbb{R}^n)$  is a vector space.

Examples. All previous examples of linear functionals are distributions.



## Locally Integrable Functions

Definition. A function  $g \colon \mathbb{R}^n \to \mathbb{C}$  such that

 $\int_{\Omega} |g(x)| \, dx < \infty \qquad \text{ for any bounded set } \Omega \subset \mathbb{R}^n$ 

is said to be locally integrable.

The space of locally integrable functions is denoted by  $L^1_{loc}(\mathbb{R}^n)$ .

Example. The following functions  $f : \mathbb{R} \to \mathbb{R}$  are locally integrable:

(i) 
$$f(x) = x^{2}$$
  
(ii)  $f(x) = H(x) = \begin{cases} 1 & x \ge 0 \\ 0 & x < 0 \end{cases}$  (Heaviside function)  
(iii)  $f(x) = \begin{cases} \ln(x) & x > 0 \\ 0 & x \le 0 \end{cases}$ 



# Regular and Singular Distributions If $g \in L^1_{loc}(\mathbb{R}^n)$ then

$$T_g: \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}, \qquad \varphi \mapsto \int_{\mathbb{R}^n} g(x)\varphi(x) \, dx$$

defines a distribution.

Definition. A distribution  $T \in \mathcal{D}'(\mathbb{R}^n)$  so that

$$T\varphi = \int_{\mathbb{R}^n} g(x)\varphi(x)\,dx$$

for some  $g \in L^1_{loc}(\mathbb{R}^n)$  is said to be a regular distribution.

A distribution that is not regular is said to be singular.



## Regular and Singular Distributions Example. The distribution $T \in \mathcal{D}'(\mathbb{R})$ given by

$$T\varphi = \int_0^\infty \varphi(x) \, dx = \int_{-\infty}^\infty H(x)\varphi(x) \, dx$$

is regular.

The Dirac delta distribution  $T_\delta \in \mathcal{D}'(\mathbb{R})$  given by

$$T_\delta \varphi := \varphi(0)$$

is singular.



#### Proof that $T_{\delta}$ is Singular

Suppose that there exists a function  $g \in L^1_{\mathsf{loc}}(\mathbb{R}^n)$  such that

$$T_{\delta}\varphi = \int_{\mathbb{R}^n} g(x)\varphi(x)\,dx = \varphi(0).$$

For a > 0 define  $\psi_a \in \mathcal{D}(\mathbb{R}^n)$ ,

$$\psi_{a}(x) = egin{cases} e^{-a^{2}/(|x|^{2}+a^{2})} & |x| < a, \ 0 & ext{otherwise} \end{cases}$$

and note

$$|\psi_a(x)| \leq \frac{1}{e}.$$



# Proof that $T_{\delta}$ is Singular

Then

$$T_{\delta}\psi_{a}| = \left| \int_{\mathbb{R}^{n}} g(x)\psi_{a}(x) \, dx \right|$$
$$\leq \frac{1}{e} \int_{|x| < a} |g(x)| \, dx$$
$$\xrightarrow{a \to 0} 0.$$

But

$$T_{\delta}\psi_{a} = \psi_{a}(0) = rac{1}{e} \not\rightarrow 0 \quad \text{as } a \rightarrow 0.$$

Contradiction!



### Outlook

Purely formally / symbolically:

$$T_{\delta}\varphi = \int_{\mathbb{R}^n} \delta(x)\varphi(x) \, dx = \varphi(0)$$
  
Dirac delta "function"

To Do:

- Prove that  $T_{\delta} / \delta(x)$  represents a "point source".
- ► Consider also "point dipoles" and similar objects.



## Outlook

#### Natural identification

$$g \in L^1_{\mathsf{loc}}(\mathbb{R}^n) \qquad \leftrightarrow \qquad T_g \in \big\{ T \in \mathcal{D}' \colon T \; \mathsf{regular} \big\}$$

leads to

$$L^1_{\mathsf{loc}}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$$

#### To Do:

- ► Extend operations of calculus (differentiation, multiplication of functions, etc.) to D'(ℝ<sup>n</sup>)
- Define convergence of sequences of distributions
- Discuss the Fourier transform
- How to solve equations in distributions?