Test Functions and Distributions

Test Functions and Linear Functionals

$\mathcal{D}(\mathbb{R}^n) := C_0^\infty(\mathbb{R}^n)$

(Space of Test Functions in $\mathbb{R}^n$)

$\mathcal{D}(\mathbb{R}^n)$ is a (complex) vector space.

Definition. A linear functional on $\mathcal{D}(\mathbb{R}^n)$ is a map

$$T\colon \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$$

such that

$$T(\lambda\varphi_1 + \mu\varphi_2) = \lambda T\varphi_1 + \mu T\varphi_2$$

for $\varphi_1, \varphi_2 \in \mathcal{D}$, $\lambda, \mu \in \mathbb{C}$.

Examples of Linear Functionals

Linear functionals on $\mathcal{D}(\mathbb{R})$:

(i) $T\varphi := \displaystyle\int_0^\infty \varphi(x)\,dx$

(ii) $T\varphi := \varphi(0)$

(iii) $T\varphi := \varphi'(1)$

Linear functionals on $\mathcal{D}(\mathbb{R}^n)$:

(i) $T\varphi := \displaystyle\int_{\mathbb{R}^n} \varphi(x)\,dx$

(ii) $T\varphi := \displaystyle\int_S \varphi\,d\sigma$, where S is a surface in $\mathbb{R}^n$

(iii) $T\varphi := \displaystyle\int_S \operatorname{grad}\varphi\,d\vec{\sigma}$

Continuous Linear Functionals

Definition. A linear functional T is said to be continuous if

$$\varphi_m \to 0 \qquad \Rightarrow \qquad T\varphi_m \to 0$$

null sequence in $\mathcal{D}(\mathbb{R}^n)$

sequence in $\mathbb{C}$

A continuous linear functional on $\mathcal{D}(\mathbb{R}^n)$ is called a distribution.

The set of all distributions is denoted by $\mathcal{D}'(\mathbb{R}^n)$.

$\mathcal{D}'(\mathbb{R}^n)$ is a vector space.

Examples. All previous examples of linear functionals are distributions.

Locally Integrable Functions

Definition. A function $g\colon \mathbb{R}^n \to \mathbb{C}$ such that

$$\int_{\Omega} |g(x)|\, dx < \infty \qquad \text{for any bounded set } \Omega \subset \mathbb{R}^n$$

is said to be locally integrable.

The space of locally integrable functions is denoted by $L^1_{\text{loc}}(\mathbb{R}^n)$.

Example. The following functions $f\colon \mathbb{R} \to \mathbb{R}$ are locally integrable:

(i) $f(x) = x^2$

(ii) $f(x) = H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ (Heaviside function)

(iii) $f(x) = \begin{cases} \ln(x) & x > 0 \\ 0 & x \leq 0 \end{cases}$

Regular and Singular Distributions

If $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ then

$$T_g : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}, \qquad \varphi \mapsto \int_{\mathbb{R}^n} g(x)\varphi(x)\,dx$$

defines a distribution.

Definition. A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ so that

$$T\varphi = \int_{\mathbb{R}^n} g(x)\varphi(x)\,dx$$

for some $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be a regular distribution.

A distribution that is not regular is said to be singular.

Regular and Singular Distributions

Example. The distribution $T \in \mathcal{D}'(\mathbb{R})$ given by

$$T\varphi = \int_0^\infty \varphi(x)\,dx = \int_{-\infty}^\infty H(x)\varphi(x)\,dx$$

is regular.

The Dirac delta distribution $T_\delta \in \mathcal{D}'(\mathbb{R})$ given by

$$T_\delta\varphi := \varphi(0)$$

is singular.

Proof that T_δ is Singular

Suppose that there exists a function $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$T_\delta \varphi = \int_{\mathbb{R}^n} g(x)\varphi(x)\,dx = \varphi(0).$$

For $a > 0$ define $\psi_a \in \mathcal{D}(\mathbb{R}^n)$,

$$\psi_a(x) = \begin{cases} e^{-a^2/(|x|^2+a^2)} & |x| < a, \\ 0 & \text{otherwise} \end{cases}$$

and note

$$|\psi_a(x)| \le \frac{1}{e}.$$

Proof that T_δ is Singular

Then

$$\begin{aligned} |T_\delta \psi_a| &= \left| \int_{\mathbb{R}^n} g(x)\psi_a(x)\,dx \right| \\ &\le \frac{1}{e} \int_{|x|<a} |g(x)|\,dx \\ &\xrightarrow{a\to 0} 0. \end{aligned}$$

But

$$T_\delta \psi_a = \psi_a(0) = \frac{1}{e} \not\to 0 \qquad \text{as } a \to 0.$$

Contradiction!

Outlook

Purely formally / symbolically:

$$T_\delta \varphi = \int_{\mathbb{R}^n} \delta(x) \varphi(x)\, dx = \varphi(0)$$

Dirac delta "function"

To Do:

- Prove that T_δ / $\delta(x)$ represents a "point source".
- Consider also "point dipoles" and similar objects.

Outlook

Natural identification

$$g \in L^1_{\text{loc}}(\mathbb{R}^n) \qquad \leftrightarrow \qquad T_g \in \{T \in \mathcal{D}' \colon T \text{ regular}\}$$

leads to

$$L^1_{\text{loc}}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$$

To Do:

- Extend operations of calculus (differentiation, multiplication of functions, etc.) to $\mathcal{D}'(\mathbb{R}^n)$
- Define convergence of sequences of distributions
- Discuss the Fourier transform
- How to solve equations in distributions?