



Null Sequences

Convergence in $C_0^\infty(\mathbb{R})$

Important for following discussion:

Sequences of smooth, compactly supported functions.

How to define convergence in $C_0^\infty(\mathbb{R})$?

Let (φ_m) be a sequence in $C_0^\infty(\mathbb{R})$. Then, given some $\varphi \in C_0^\infty(\mathbb{R})$,

$$\varphi_m \rightarrow \varphi \quad \text{if and only if} \quad \underbrace{\varphi_m - \varphi}_{=:\psi_m} \rightarrow 0.$$

It is sufficient to define what it means that a sequence (ψ_m) converges to zero. Such a sequence will be called a **null sequence**.

Null Sequences

Definition (φ_m) is a **null sequence** in $C_0^\infty(\mathbb{R}^n)$ if

- (i) There exists some $R > 0$ such that for all $m \in \mathbb{N}$,

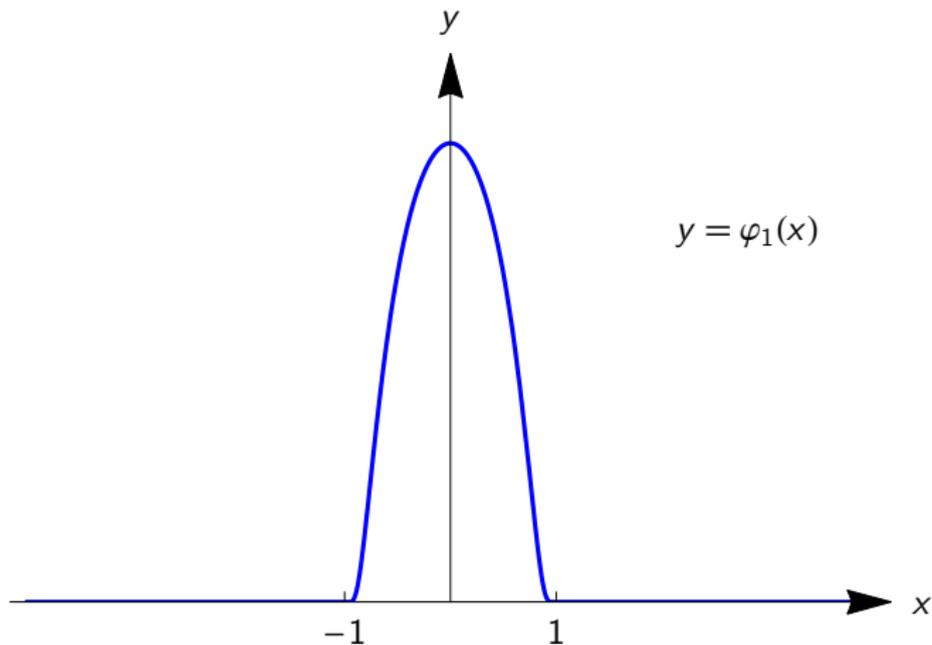
$$\text{supp } \varphi_m \subset B_R(0) = \{x \in \mathbb{R}^n : |x| < R\}.$$

- (ii) For every multi-index $\alpha \in \mathbb{N}^n$,

$$\sup_{x \in \mathbb{R}^n} |D^\alpha \varphi_m(x)| \xrightarrow{m \rightarrow \infty} 0.$$

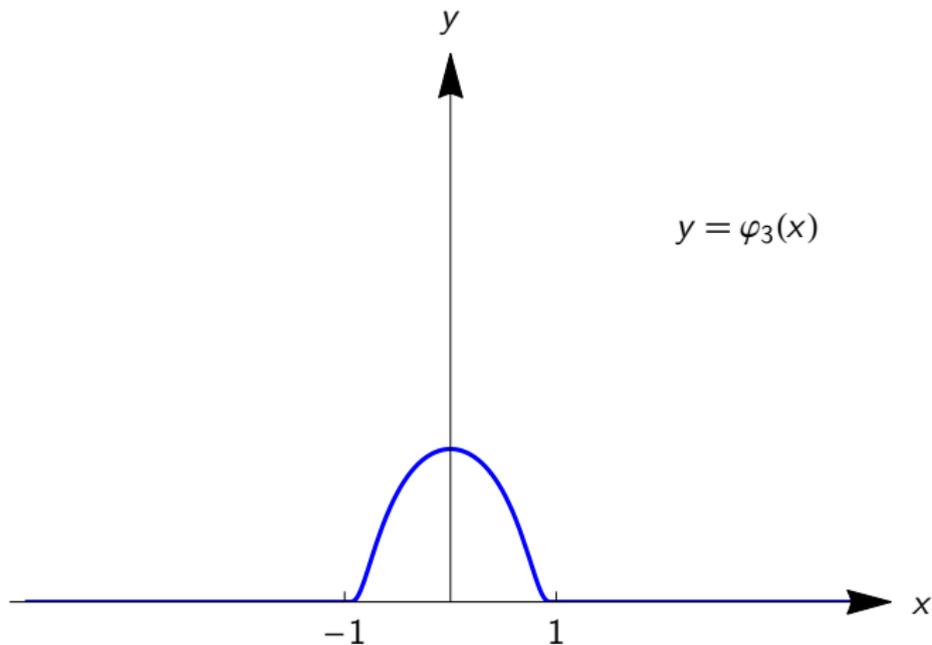
Example of a Null Sequence

Let $\varphi \in C_0^\infty(\mathbb{R})$ and define $\varphi_m(x) := \varphi(x)/m$.



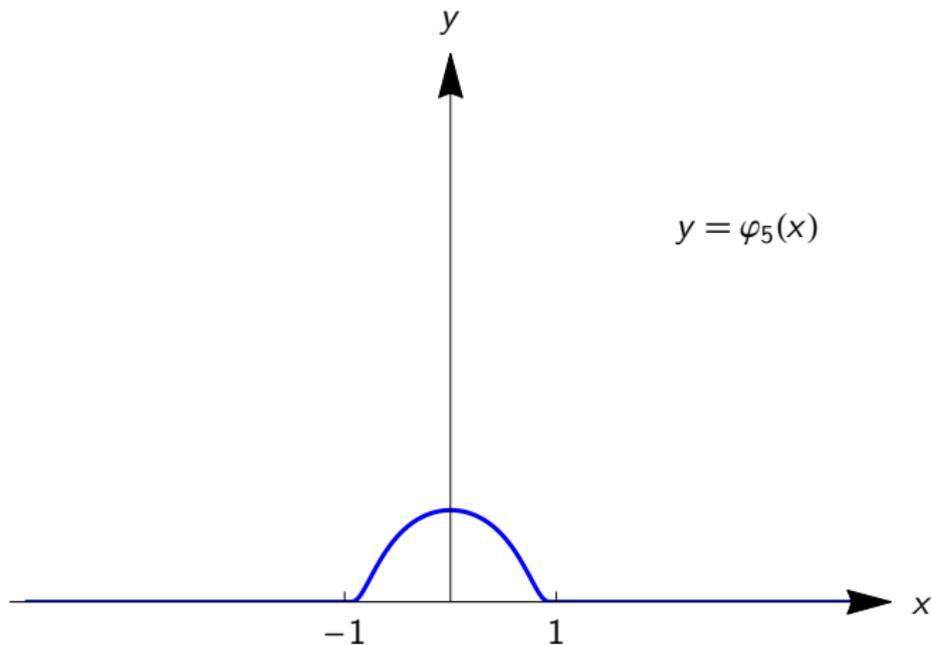
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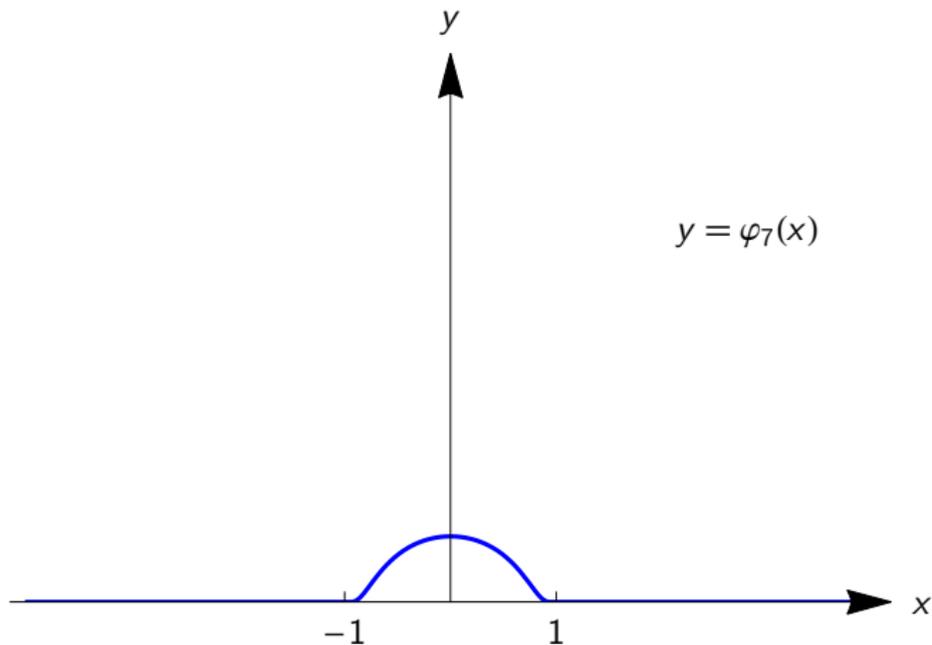
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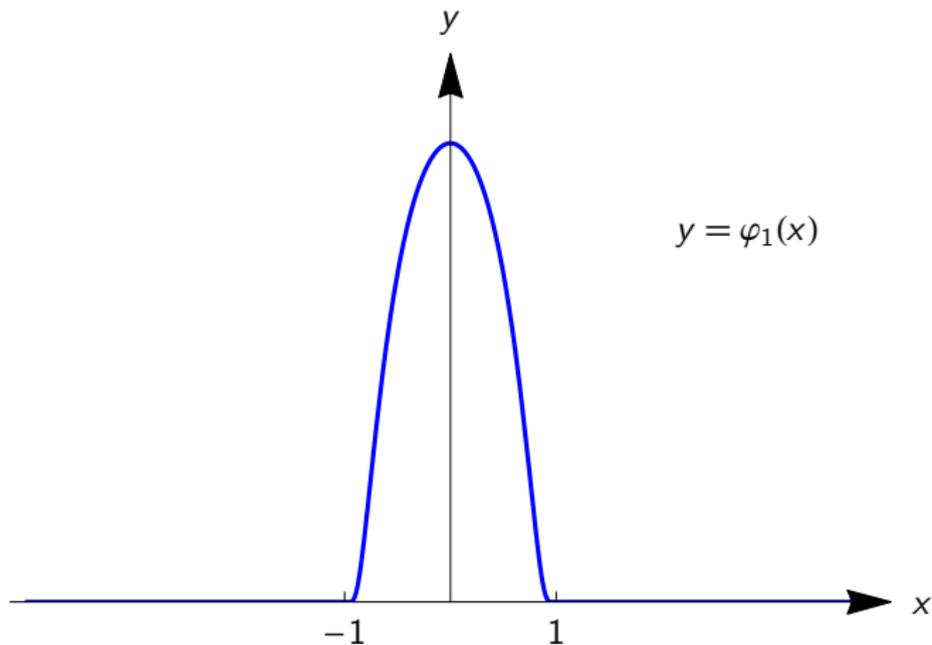
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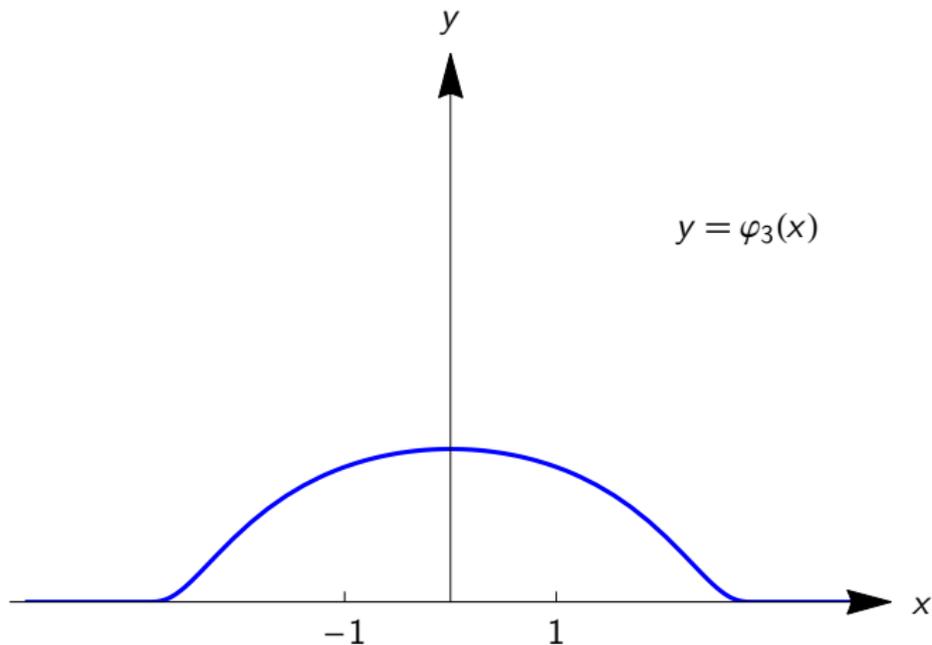
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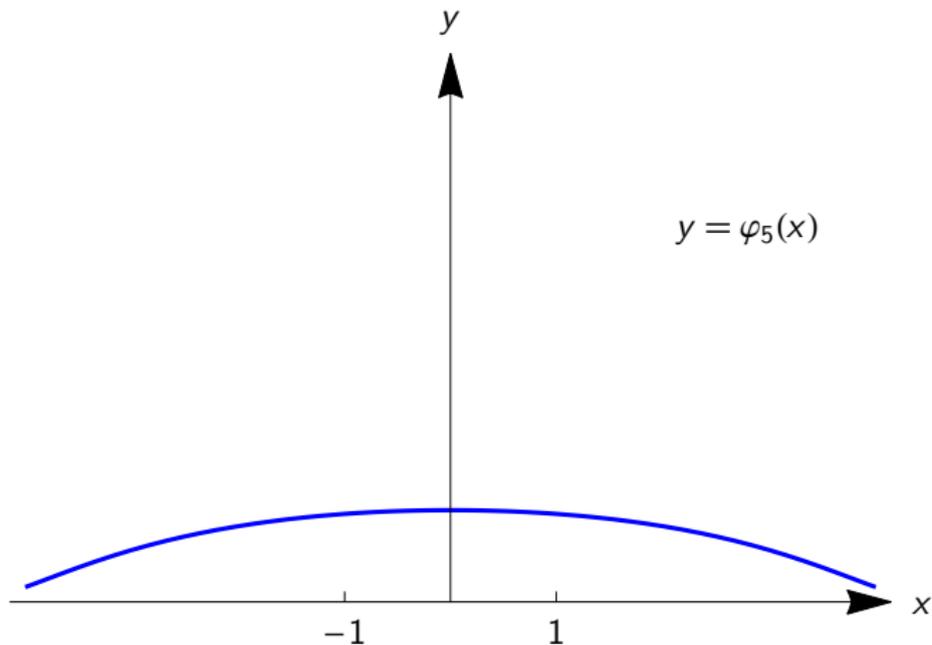
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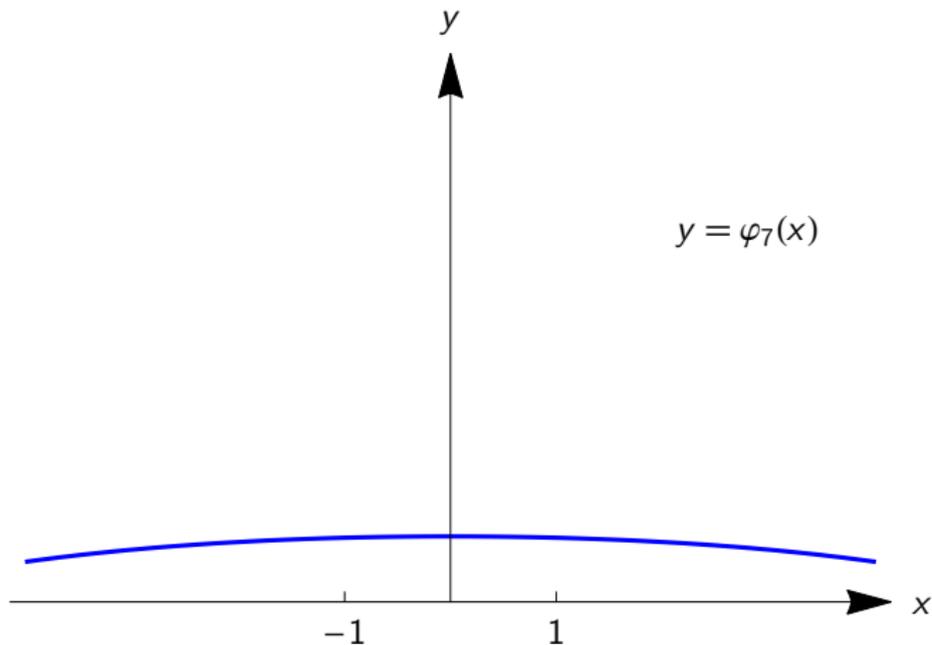
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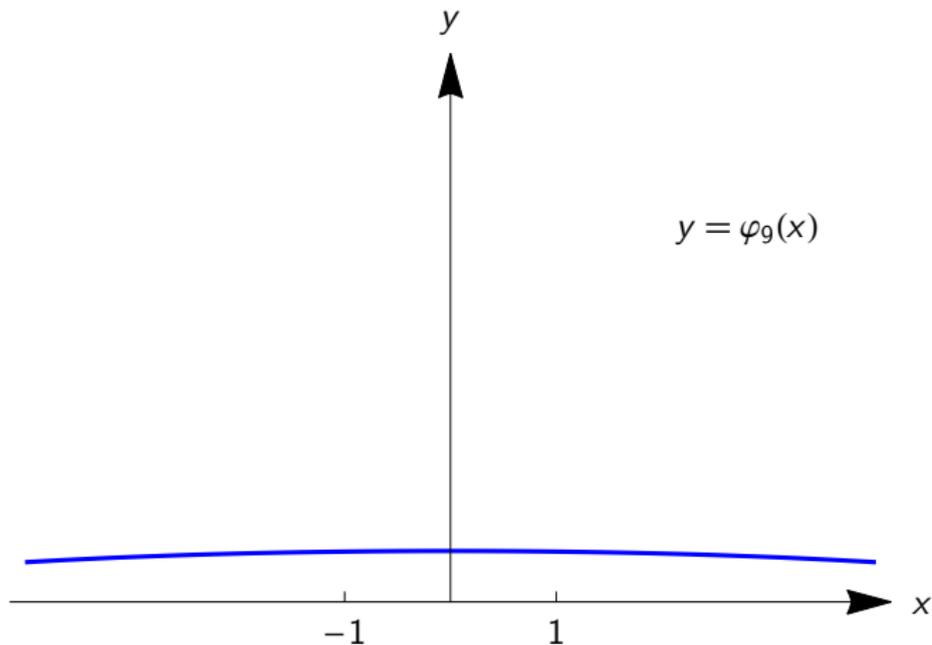
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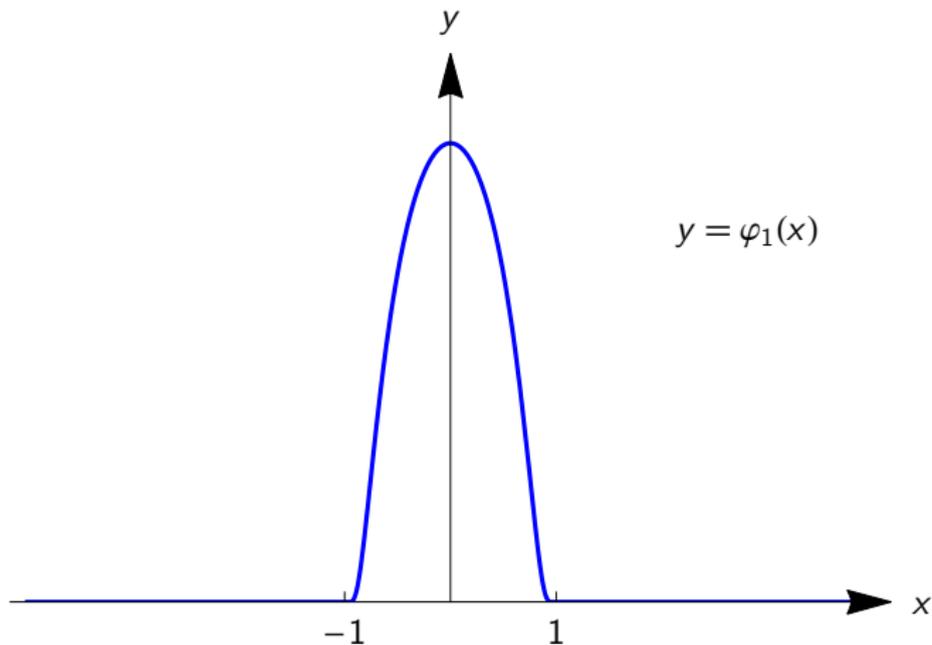
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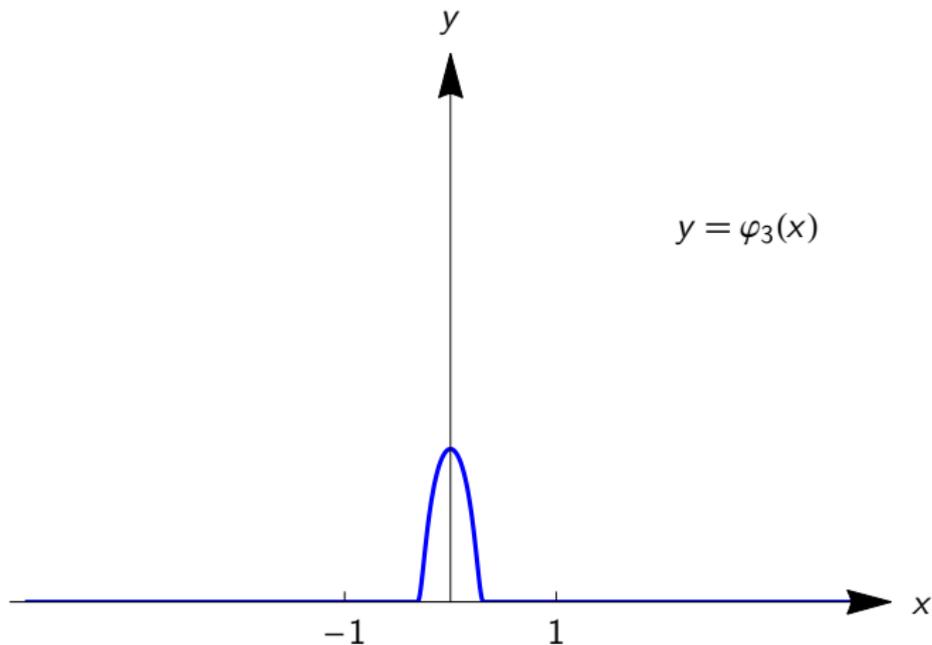
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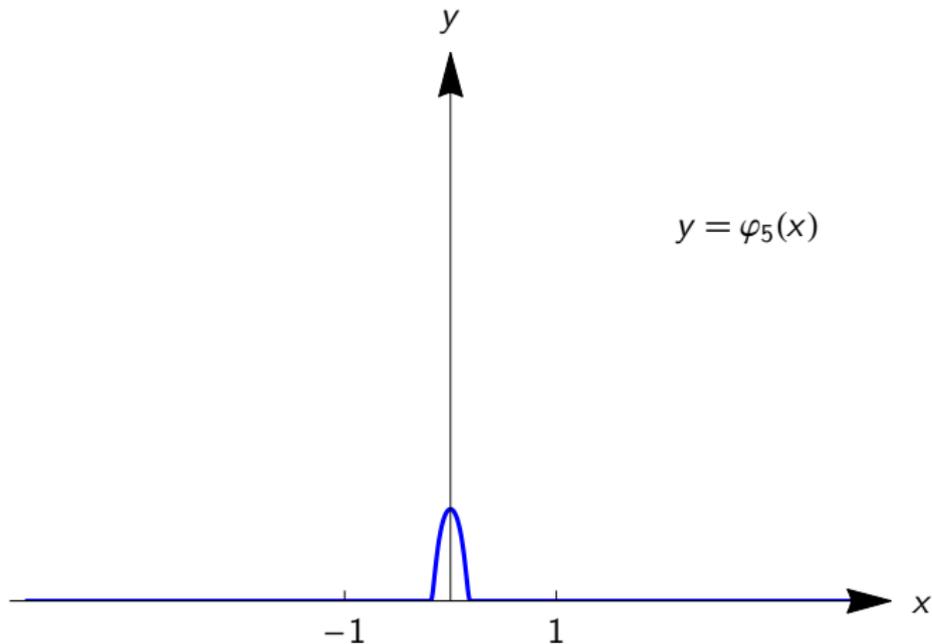
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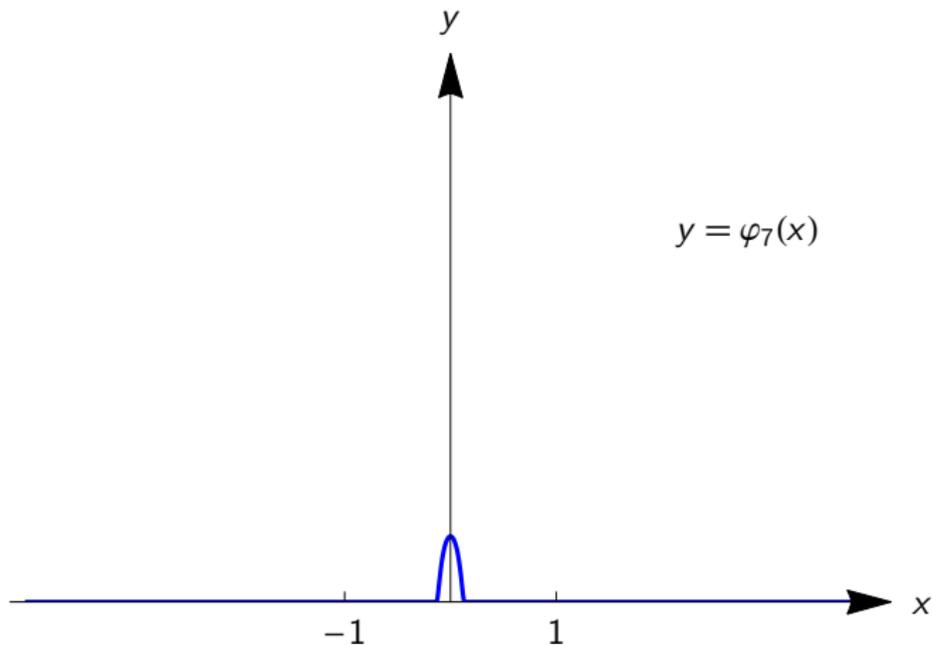
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