Smooth, Compactly Supported Functions

Smooth Functions

Definitions:

- A domain is an open and simply connected set in $\mathbb{R}^n$. We reserve the symbol Ω for domains.
- The set of k times continuously differentiable functions on a domain Ω:

$$C^k(\Omega) := \{\varphi \colon \Omega \to \mathbb{C} \colon \text{all partial derivatives of } \varphi \\ \text{of order } k \text{ exist and are continuous}\}.$$

- The set of smooth functions on Ω:

$$C^\infty(\Omega) := \{\varphi \colon \Omega \to \mathbb{C} \colon \text{all partial derivatives of } \varphi \\ \text{of any order exist}\}.$$

Compactly Supported Functions

- The support of a function $\varphi\colon \Omega \to \mathbb{C}$:

$$\operatorname{supp}\varphi := \overline{\{x \in \Omega \colon \varphi(x) \neq 0\}}.$$

  ($\overline{A}$ denotes the closure of a set $A \subset \mathbb{R}^n$.)

- The set of smooth functions with compact support:

$$C_0^\infty(\mathbb{R}^n) := \{\varphi \in C^\infty(\mathbb{R}^n) \colon \operatorname{supp}\varphi \text{ is a bounded set}\}$$

$$C_0^\infty(\Omega) := \{\varphi \in C_0^\infty(\mathbb{R}^n) \colon \operatorname{supp}\varphi \subset \Omega\}$$

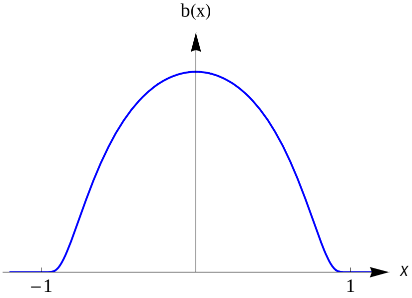
Problem: Do there even exist functions in $C_0^\infty(\mathbb{R}^n)$?

The Bump Function

Define the bump function

$$b\colon \mathbb{R} \to \mathbb{R}, \qquad b(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & -1 \le x \le 1, \\ 0 & |x| > 1. \end{cases}$$

(For $\mathbb{R}^n$, consider simply $b(|x|)$.)

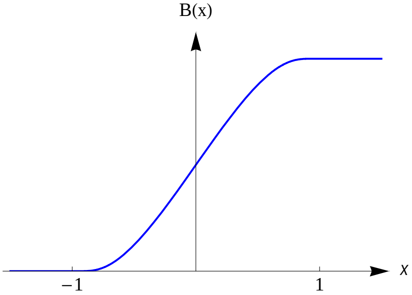


The Smooth Step

The integral of the bump function,

$$B(x) := \int_{-\infty}^{x} b(y)\, dy,$$

creates a “smooth step”:



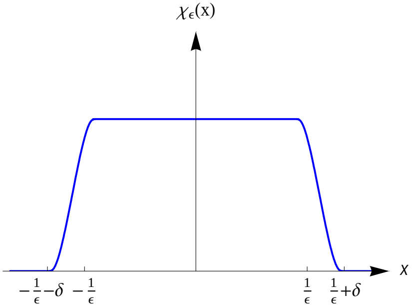
Cut-Off Functions

Shift and scale B to create a cut-off function χ_ε with the properties

(i) $\chi_\varepsilon \in C_0^\infty(\mathbb{R})$,

(ii) $\chi_\varepsilon(x) = 1$ if $|x| < 1/\varepsilon$,

(iii) $\chi_\varepsilon(x) = 0$ if $|x| > 1/\varepsilon + \delta$ (here $\delta > 0$ may depend on ε)



Constructing Smooth, Compactly Supported Functions

If $f \in C^\infty(\mathbb{R})$, the function f_ε defined by

$$f_\varepsilon(x) := \chi_\varepsilon(x) \cdot f(x)$$

satisfies

$$f_\varepsilon(x) = \begin{cases} f(x) & \text{for } |x| < 1/\varepsilon, \\ 0 & \text{for } |x| > 1/\varepsilon + \varepsilon. \end{cases}$$

Furthermore, $f_\varepsilon \in C^\infty(\mathbb{R})$ since both χ_ε and f are smooth functions. Hence,

$$f_\varepsilon \in C_0^\infty(\mathbb{R}).$$

Conclusion: There are many functions in $C_0^\infty(\mathbb{R})$!

Multi-Index Notation for Smooth Functions

A multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is an n-tuple of natural numbers. We define

- Degree of α:

$$|\alpha| = \alpha_1 + \cdots + \alpha_n$$

- Derivatives:

$$D^\alpha u := \frac{\partial^\alpha u}{\partial x^\alpha} := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$

- Monomials:

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

- Factorials:

$$\alpha! := \alpha_1! \alpha_2! \cdots \alpha_n!$$