Further Approaches to the Green Function

Eigenfunction Expansion of the Green Function

Different approach to the Green function for the problem

$$-\frac{d^2u}{dx^2} = f(x), \qquad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

Consider the associated eigenvalue problem

$$-u''(x) = \lambda u, \qquad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

One finds

- eigenvalues $\lambda_n = n^2\pi^2$, $n \in \mathbb{N} \setminus \{0\}$
- eigenfunctions $u_n(x) = \sin(n\pi x)$

Eigenfunction Expansion of Green's Function

Multiply $-u'' = f$ with u_n and integrate:

$$-\int_0^1 u_n(x)u''(x)\,dx = \int_0^1 f(x)u_n(x)\,dx.$$

Integrate the LHS by parts twice, use $-u_n'' = \lambda_n u_n$:

$$\int_0^1 u_n(x)u(x)\,dx = \frac{1}{\lambda_n}\int_0^1 f(x)u_n(x)\,dx.$$

Now expand u into a Fourier-sine series:

$$\begin{aligned} u(x) &= \sum_{n=1}^{\infty} 2\int_0^1 u_n(\xi)u(\xi)\,d\xi\ \sin(n\pi x) \\ &= \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\int_0^1 f(\xi)u_n(\xi)\,d\xi\ \sin(n\pi x). \end{aligned}$$

Eigenfunction Expansion of the Green Function

Interchange the infinite series with the integral:

$$u(x) = \int_0^1 g(x,\xi) f(\xi)\, d\xi$$

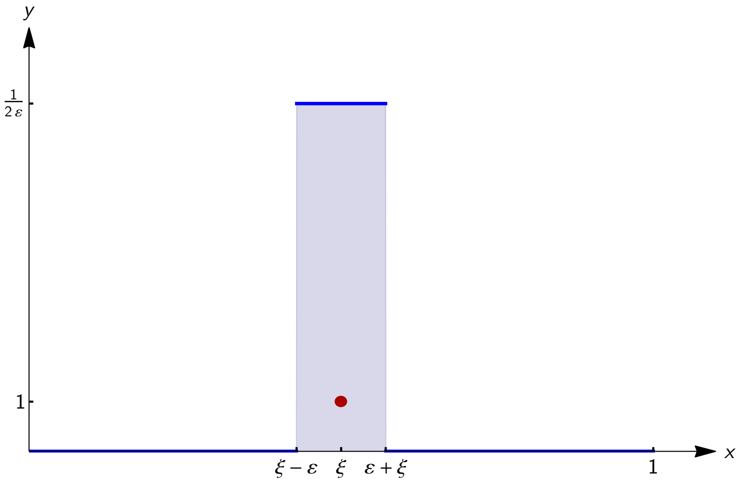
where

$$g(x,\xi) = \sum_{n=1}^{\infty} \frac{2\sin(n\pi x)\sin(n\pi\xi)}{n^2\pi^2}.$$

(Eigenfunction Expansion of the Green function)

Physical Model of a Point Source

Heat $Q(t) = 1$ is generated uniformly in the interval $[\xi - \varepsilon, \xi + \varepsilon]$



Physical Approach to the Green Function

Then $Q = \int_0^1 q_\varepsilon(x;\xi)\,dx$ where

$$q_\varepsilon(x;\xi) = \begin{cases} 0 & |x-\xi| > \varepsilon, \\ \frac{1}{2\varepsilon} & |x-\xi| \le \varepsilon. \end{cases}$$

Of course, $\lim\limits_{\varepsilon \to 0} q_\varepsilon(x;\xi)$ does not exist!

But: For any $\varepsilon > 0$ there exists a classical solution $u_\varepsilon(x,\xi)$ of

$$-u'' = q_\varepsilon(x;\xi), \qquad u(0) = u(1) = 0.$$

It can be shown that

$$g(x,\xi) = \lim_{\varepsilon \to 0} u_\varepsilon(x,\xi).$$

The Dirac Delta "Function" δ_ξ

Symbolically, we want to write

$$-g''(x,\xi) = \delta_\xi(x),$$

where

$$\delta_\xi(x) :\overset{?}{=} \lim_{\varepsilon \to 0} q_\varepsilon(x;\xi).$$

Necessary Properties of δ_ξ

(i) $\delta_\xi(x) = 0$ if $x \neq \xi$ (there is no heat source at $x \neq \xi$)

(ii) $\int_0^1 \delta_\xi(x)\,dx = 1$ (the total heat generated is equal to one)

But such a function does not exist!

Goals for this Course

- Define generalized functions, that include objects like $\delta_\xi(x)$
- Extend calculus to generalized functions.
  Upshot: (nearly) all functions can be differentiated.
- Define non-classical solutions to differential equations.
- Formalize the concept of Green functions and their use in solving differential equations.
- Investigate methods for finding Green functions.
- Introduce a numerical method that uses Green functions.