Proof of the Solution Formula

The Equilibrium Heat Equation

Theorem.

Let $\alpha, \beta \in \mathbb{R}$ and $f \in C([0,1])$ be given. Then the unique solution of

$$-u''(x) = f(x), \qquad 0 < x < 1, \qquad u(0) = \alpha, \qquad u(1) = \beta$$

is given by

$$u(x; \alpha, \beta) = \int_0^1 f(\xi) g(x, \xi)\, d\xi + \alpha(1 - x) + \beta x.$$

Existence of a Solution

Existence. We write

$$g(x,\xi) = \begin{cases} (1-\xi)x & 0 \le x < \xi \\ (1-x)\xi & \xi \le x \le 1 \end{cases}$$

$$= \begin{cases} l(x,\xi) & 0 \le x < \xi, \\ r(x,\xi) & \xi \le x \le 1. \end{cases}$$

Since

$$g(0,\xi) = g(1,\xi) = 0$$

it is sufficient to consider $\alpha = \beta = 0$.

Existence of a Solution

The derivative of $u(x;0,0)$ is given by

$$\frac{du}{dx} = \frac{d}{dx}\int_0^x f(\xi)r(x,\xi)\,d\xi + \frac{d}{dx}\int_x^1 f(\xi)l(x,\xi)\,d\xi$$

By the chain rule

$$\begin{aligned}\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} h(x,y)\,dy = \int_{\alpha(x)}^{\beta(x)} \frac{dh}{dx}(x,y)\,dy \\ + \beta'(x)h(x,\beta(x)_-) - \alpha'(x)h(x,\alpha(x)_+).\end{aligned}$$

where

$$f(x_\pm) := \lim_{\varepsilon\to 0} f(x\pm\varepsilon)$$

for any function f.

Existence of a Solution

We hence obtain

$$\frac{du}{dx} = \int_0^x f(\xi) r_x(x,\xi)\, d\xi + \int_x^1 f(\xi) l_x(x,\xi)\, d\xi \\ + r(x,x_-)f(x_-) - l(x,x_+)f(x_+)$$

Since f and g are continuous,

$$f(x_-) = f(x_+) \qquad \text{and} \qquad r(x,x_-) = l(x,x_+),$$

so

$$\frac{du}{dx} = \int_0^x f(\xi) r_x(x,\xi)\, d\xi + \int_x^1 f(\xi) l_x(x,\xi)\, d\xi$$

Existence of a Solution

Differentiating

$$\frac{du}{dx} = \int_0^x f(\xi) r_x(x,\xi)\, d\xi + \int_x^1 f(\xi) l_x(x,\xi)\, d\xi$$

once more:

$$\begin{aligned}\frac{d^2u}{d^2x} &= \int_0^x f(\xi) r_{xx}(x,\xi)\, d\xi + \int_x^1 f(\xi) l_{xx}(x,\xi)\, d\xi \\ &\quad + f(x)\big(r_x(x,x_-) - l_x(x,x_+)\big) \\ &= -f(x).\end{aligned}$$

This proves that a solution exists and is given by

$$u(x;\alpha,\beta) = \int_0^1 f(\xi) g(x,\xi)\, d\xi + \alpha(1-x) + \beta x.$$

Uniqueness of the Solution

Uniqueness. Suppose that u_1 and u_2 are two classical solutions of

$$-u''(x) = f(x), \qquad 0 < x < 1, \qquad u(0) = \alpha, \qquad u(1) = \beta$$

Then

$$v = u_1 - u_2$$

is twice continuously differentiable and satisfies

$$-v''(x) = 0, \qquad 0 < x < 1, \qquad v(0) = 0, \qquad v(1) = 0$$

Uniqueness of the Solution

In particular, $v''(x) = 0$ for all $x \in (0,1)$ and

$$v(x) = Ax + B \qquad \text{for } 0 < x < 1 \text{ and some } A, B \in \mathbb{R}.$$

Since $v(0) = v(1) = 0$ and v is continuous on $[0,1]$,

$$v(x) = 0$$

for all $x \in [0,1]$.

Hence,

$$u_1 = u_2.$$

This proves that there exists only one solution.