The Boundary Element Method

The Boundary Element Method (BEM)

Step 1: Approximate Ω by a polygon

- Choose $x^{(k)} \in \partial\Omega$, $k = 1, \ldots, N$, and join by straight line segments (Boundary elements).

The line element joining $x^{(k)}$ to $x^{(k+1)}$ is denoted $\mathcal{C}^{(k)}$ ($x^{(N+1)} := x^{(1)}$).

Step 2: Approximate boundary data

- Find midpoint $\overline{x}^{(k)}$ of $\mathcal{C}^{(k)}$
- Take

$$u\big|_{\mathcal{C}^{(k)}} \approx \overline{u}^{(k)} = u(x^{(k)}) \qquad \text{on } S_1$$

$$\frac{\partial u}{\partial n}\Big|_{\mathcal{C}^{(k)}} \approx \overline{p}^{(k)} = \frac{\partial u}{\partial n}\Big|_{x^{(k)}} \qquad \text{on } S_2$$

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Then

$$\lambda_{\Omega}(\xi)u(\xi) \approx \sum_{k=1}^{N} \overline{u}^{(k)} \cdot I_2^{(k)}(\xi) - \overline{p}^{(k)} I_1^{(k)}(\xi).$$

with

$$I_1^{(k)}(\xi) := \int_{\mathcal{C}^{(k)}} E(\,\cdot\,;\xi)\,ds, \qquad I_2^{(k)}(\xi) := \int_{\mathcal{C}^{(k)}} \frac{\partial E(\,\cdot\,;\xi)}{\partial n}\,ds.$$

Easily calculated!

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Choose $\xi = \overline{x}^{(k)}$:

$$\frac{1}{2}\underbrace{u(\overline{x}^{(k)})}_{=\overline{u}^{(k)}} = \sum_{k=1}^{N} \overline{u}^{(k)} \cdot I_2^{(k)}(\overline{x}^{(k)}) - \overline{p}^{(k)} I_1^{(k)}(\overline{x}^{(k)})$$

- Linear system of N algebraic equations
- $2N$ unknowns $\overline{u}^{(k)}$ and $\overline{p}^{(k)}$, $k = 1, \ldots, N$
- N unknowns given by boundary data

$\Rightarrow$ Find all coefficients $\overline{u}^{(k)}$ and $\overline{p}^{(k)}$.

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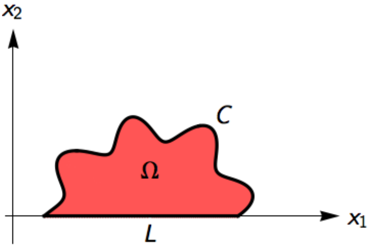
Then

$$u(\xi) \approx \sum_{k=1}^{N} \overline{u}^{(k)} \cdot I_2^{(k)}(\xi) - \overline{p}^{(k)} I_1^{(k)}(\xi)$$

for $\xi \in \Omega$.

Green Functions and the BEM

Suppose $\partial\Omega = L \cup \mathcal{C}$, $\mathcal{C} = S_1 \cup S_2$.



Require

$$u\big|_L = 0$$

and

$$u\big|_{S_1} = f, \qquad \frac{du}{dn}\bigg|_{S_2} = g,$$

Use Green's function for upper half-plane,

$$g(x;\xi) = E(x,\xi) - E(x;\xi^*)$$

with $\xi^* = (\xi_1, -\xi_2)$

Green Functions and the BEM

We find

$$\lambda_{\Omega'}(\xi)u(\xi) = \int_{\mathcal{C}} \Big(u \cdot \frac{\partial g(\,\cdot\,;\xi)}{\partial n} - g(\,\cdot\,;\xi) \cdot \frac{\partial u}{\partial n} \Big)\, ds$$

where

$$\lambda_{\Omega'}(\xi) = \begin{cases} 0 & \xi \in L \cup \overline{\Omega}^c, \\ 1 & \xi \in \Omega, \\ 1/2 & \xi \in \mathcal{C}. \end{cases}$$

Note: We do not need to integrate over L, as

$$u\big|_L = g\big|_L = 0$$

Green Functions and the BEM

Discretize $\mathcal{C}$: $x^{(1)}$, $x^{(N+1)}$ are the endpoints of $\mathcal{C}$.

Then

$$\lambda_\Omega(\xi)u(\xi) \approx \sum_{k=1}^{N} \overline{u}^{(k)} \cdot \int_{\mathcal{C}^{(k)}} \frac{\partial g}{\partial n}(\,\cdot\,;\xi)\,ds - \overline{p}^{(k)} \int_{\mathcal{C}^{(k)}} g(\,\cdot\,;\xi)\,ds.$$

where

$$\int_{\mathcal{C}^{(k)}} g(\,\cdot\,;\xi)\,ds = I_1^{(k)}(\xi) - I_1^{(k)}(\xi^*),$$
$$\int_{\mathcal{C}^{(k)}} \frac{\partial g}{\partial n}(\,\cdot\,;\xi)\,ds = I_2^{(k)}(\xi) - I_2^{(k)}(\xi^*).$$

As before, find the N unknown parameters of $\overline{u}^{(k)}$ and $\overline{p}^{(k)}$.

Green Functions and the BEM

Advantage of Green functions in BEM:

- Smaller part of $\partial\Omega$ to discretize, fewer equations / unknowns.

Disdvantage of Green functions in BEM:

- The integrals

$$\int_{\mathcal{C}^{(k)}} g(\,\cdot\,;\xi)\,ds \qquad \text{and} \qquad \int_{\mathcal{C}^{(k)}} \frac{\partial g}{\partial n}(\,\cdot\,;\xi)\,ds$$

  may be harder to evaluate