Exploiting Symmetries for Image Charges

The Dirichlet Problem on the Unit Disk

Method of images exploits symmetry of a domain.

Discuss examples in $\mathbb{R}^2$ for $L = -\Delta$ using

$$E(x; \xi) = \frac{1}{2\pi} \ln|x - \xi|$$

Consider the unit disk

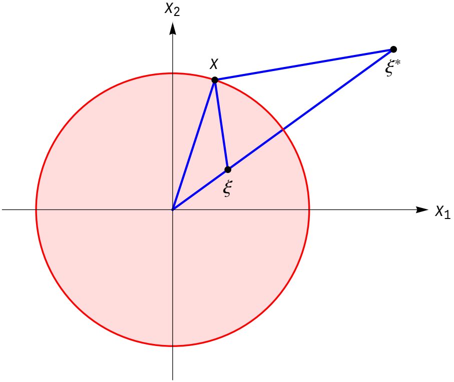
$$D = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 + x_2^2 \leq 1\}.$$

For $\xi \in D$ set

$$\xi^* = \frac{\xi}{|\xi|^2}.$$

The Dirichlet Problem on the Unit Disk

For $\xi \in D$ set $\xi^* = \dfrac{\xi}{|\xi|^2}$.



$\triangle(0, x, \xi)$ is similar to $\triangle(0, x, \xi^*)$

Therefore,

$$\frac{|x - \xi|}{|\xi|} = \frac{|x - \xi^*|}{|x|}.$$

Since $|x| = 1$,

$$|x - \xi| = |\xi| \cdot |x - \xi^*|.$$

The Dirichlet Problem on the Unit Disk

Algebraically:

$$\begin{aligned} |x-\xi|^2 &= 1 + |\xi|^2 - 2\langle x, \xi\rangle \\ &= \left\langle |\xi|x - \frac{\xi}{|\xi|}, |\xi|x - \frac{\xi}{|\xi|} \right\rangle \\ &= |\xi|^2 \cdot |x - \xi^*|^2 \end{aligned}$$

It follows that we can take Green's function to be

$$\begin{aligned} g(x;\xi) &= E(x;\xi) - \frac{1}{2\pi}\ln\big(|\xi|\cdot|x-\xi^*|\big) \\ &= E(x;\xi) - E(x;\xi^*) - \frac{1}{2\pi}\ln(|\xi|). \end{aligned}$$

The Dirichlet Problem on the Unit Disk

The Dirichlet problem

$$\Delta u = 0, \qquad x \in D, \qquad\qquad u\big|_{\partial D} = h$$

then has the solution

$$u(r\cos\theta, r\sin\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)K(r,\theta;a,\varphi)\big|_{a=1}\,d\varphi$$

(Poisson's integral formula)

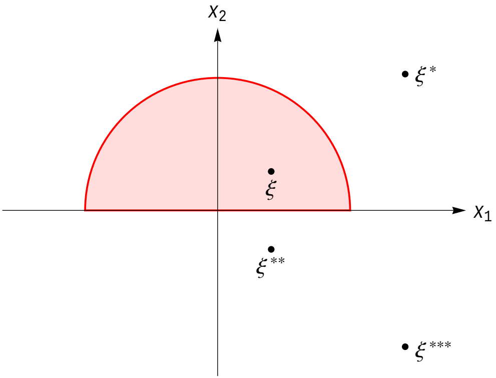
$$K(r,\theta;a,\varphi) = \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\varphi) + r^2}.$$

(Poisson kernel)

The Dirichlet Problem on the Semi-Disk

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 + x_2^2 < 1,\ x_2 > 0\}$$

Place image charges as follows:



The Dirichlet Problem on the First Quadrant

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > 0\}$$

Place image charges as follows: