The Method of Images

The Method of Images

Wanted: Green function satisfying

$$Lg(x;\xi) = \delta(x-\xi), \qquad x,\xi \in \Omega \subset \mathbb{R}^n, \qquad Bg = 0.$$

Standard Approach: Let $E(x;\xi)$ be a fundamental solution "with pole at ξ " and set

$$g(x;\xi) = E(x;\xi) + v(x)$$

where v satisfies

$$Lv = 0, \qquad x \in \Omega \subset \mathbb{R}^n, \qquad Bv = -BE(\,\cdot\,,\xi).$$

The Method of Images

Method of Images: Use the fundamental solution E to construct v, exploiting the symmetry of Ω.

Basic idea:

$$LE(x;\xi^*) = 0, \qquad x \in \Omega, \qquad \xi^* \notin \Omega$$

Choose ξ^* to satisfy the boundary conditions for the original problem.

Green's Function for $L = -\Delta$ on the Half-Plane

Half-Plane: $\mathbb{H} := \{(x_1, x_2) \in \mathbb{R}^2 \colon x_2 > 0\}$

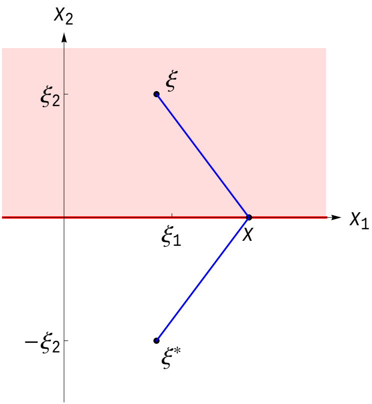
Boundary: $\partial\mathbb{H} = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_2 = 0\}$

Dirichlet Conditions: $g|_{\partial\mathbb{H}} = 0$

Fundamental Solution: $E(x;\xi) = -\frac{1}{2\pi} \ln|x - \xi|$

$$-\Delta E = \delta(x - \xi)$$

Green's Function for the Half-Plane



For $\xi = (\xi_1, \xi_2) \in \mathbb{H}$ set

$$\xi^* := (\xi_1, -\xi_2) \notin \mathbb{H}.$$

Then

$$|x - \xi| = |x - \xi^*| \quad \text{for } x \in \partial\mathbb{H}$$

and

$$g(x;\xi) = E(x;\xi) - E(x;\xi^*)$$

will vanish for $x \in \partial\mathbb{H}$.

The Dirichlet Problem on the Half-Plane

For example, the solution to the Dirichlet problem

$$\begin{aligned} \Delta u &= 0, && x \in \mathbb{H}, \\ u(x_1, 0) &= h(x_1), && x_1 \in \mathbb{R}, \quad h \in L^1_{\text{loc}}(\mathbb{R}) \end{aligned}$$

is given by

$$u(x) = \int_{\partial\mathbb{H}} h \cdot \frac{\partial g}{\partial n}\, ds = -\int_{\mathbb{R}} h(\xi_1) \cdot \frac{\partial g(x; \xi_1, \xi_2)}{\partial \xi_2}\bigg|_{\xi_2=0} d\xi_1.$$

An easy calculation yields

$$\frac{\partial g(x; \xi_1, \xi_2)}{\partial \xi_2}\bigg|_{\xi_2=0} = -\frac{1}{\pi}\frac{x_2}{x_2^2 + (x_1 - \xi_1)^2}$$

The Dirichlet Problem on the Half-Plane

Solution formula:

$$u(x_1, x_2) = \frac{1}{\pi}\int_{-\infty}^{\infty} h(y)\frac{x_2}{x_2^2 + (x_1 - y)^2}\, dy.$$

- Check directly that u satisfies the equation
- Check that u satisfies boundary conditions:

$$\begin{aligned} u(x_1, x_2) &= \frac{1}{\pi}\int_{-\infty}^{\infty} h(y)\frac{x_2}{x_2^2 + (x_1 - y)^2}\, dy \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} h(y + x_1)\frac{x_2}{x_2^2 + y^2}\, dy \quad \xrightarrow{x_2 \to 0} \quad h(x_1) \end{aligned}$$

since $\dfrac{1}{\pi}\dfrac{x_2}{x_2^2 + y^2}$ is a delta family as $x_2 \to 0$.