Partial Eigenfunction Expansions

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Full eigenfunction expansion: involves multiple series,

- unwieldy and difficult to evaluate approximately

Partial eigenfunction expansion: involves fewer series.

Strategy ($\Omega \subset \mathbb{R}^2$):

- Separate variables and solve eigenvalue problem for each variable x_1 and x_2
- Expand Green function in terms of x_1 or x_2 eigenfunctions
- Find coefficients by solving a Green's function problem for an ODE

Example: Dirichlet Problem on a Rectangle

Wanted: Green function g satisfying

$$-\Delta g(x;\xi) = \delta(x-\xi), \qquad x \in \Omega = (0,a)\times(0,b),$$

with Dirichlet conditions

$$g(x;\xi)\big|_{x_1=0} = g(x;\xi)\big|_{x_1=a} = 0, \qquad x_2 \in [0,b],$$
$$g(x;\xi)\big|_{x_2=0} = g(x;\xi)\big|_{x_2=b} = 0, \qquad x_1 \in [0,a],$$

for fixed $\xi \in \Omega$.

Step 1: Separation of Variables

Formally solve

$$-\Delta u = 0 \qquad \text{on } \Omega, \qquad\qquad u\big|_{\partial\Omega} = 0$$

by setting

$$u(x_1, x_2) = X_1(x_1) X_2(x_2).$$

Eigenvalue problems:

$$\begin{aligned} X_1'' &= -\lambda X_1, && 0 < x_1 < a, && X_1(0) = X_1(a) = 0, \\ X_2'' &= \lambda X_2, && 0 < x_2 < b, && X_2(0) = X_2(b) = 0. \end{aligned}$$

Orthonormal eigenfunctions and eigenvalues

$$\left\{ \sqrt{\frac{2}{a}} \sin\left(\frac{m\pi x_1}{a}\right) \right\}_{m=1}^{\infty}, \qquad\qquad \lambda_m = \left(\frac{m\pi}{a}\right)^2$$

Step 2: Expand the Green function

Fix (for example) $x_2 \in [0, b]$.

Expand g in terms of the x_1-eigenfunctions:

$$g(x_1, x_2; \xi) = \sum_{m=1}^{\infty} g_m(x_2; \xi) \sin\left(\frac{m\pi x_1}{a}\right).$$

Here

$$g_m(x_2; \xi) = \frac{2}{a} \int_0^a g(x; \xi) \sin\left(\frac{m\pi x_1}{a}\right)\, dx_1.$$

Step 3: Determine the Coefficients

$$\int_0^a \frac{2}{a} \sin\left(\frac{m\pi x_1}{a}\right) (-\Delta) g(x;\xi)\, dx_1 = \frac{2}{a} \sin\left(\frac{m\pi \xi_1}{a}\right) \delta(x_2 - \xi_2)$$

Writing out the left-hand side,

$$\begin{aligned}
&- \int_0^a \frac{2}{a} \sin\left(\frac{m\pi x_1}{a}\right) \Delta g(x;\xi)\, dx_1 \\
&= - \int_0^a \frac{2}{a} \sin\left(\frac{m\pi x_1}{a}\right) \frac{\partial^2 g}{\partial x_1^2}\, dx_1 - \int_0^a \frac{2}{a} \sin\left(\frac{m\pi x_1}{a}\right) \frac{\partial^2 g}{\partial x_2^2}\, dx_1 \\
&= \frac{m^2\pi^2}{a^2} g_m(x_2;\xi) - \frac{\partial^2 g_m}{\partial x_2^2}
\end{aligned}$$

Step 3: Determine the Coefficients

Using the boundary conditions on Ω, g_m satisfies

$$\frac{\partial^2 g_m}{\partial x_2^2} - \frac{m^2\pi^2}{a^2} g_m(x_2;\xi) = -\frac{2}{a}\sin\left(\frac{m\pi\xi_1}{a}\right)\delta(x_2-\xi_2)$$

for $0 < x_2 < b$, with

$$g_m(0;\xi) = g_m(b;\xi) = 0.$$

This is a Green function problem for an ODE!

We obtain

$$g_m(x_2;\xi) = \begin{cases} \frac{2}{m\pi}\frac{\sin(m\pi\xi_1/a)}{\sinh(m\pi b/a)}\sinh(\frac{m\pi x_2}{a})\sinh(\frac{m\pi(b-\xi_2)}{a}), & x_2 < \xi_2, \\ \frac{2}{m\pi}\frac{\sin(m\pi\xi_1/a)}{\sinh(m\pi b/a)}\sinh(\frac{m\pi\xi_2}{a})\sinh(\frac{m\pi(b-x_2)}{a}), & x_2 > \xi_2. \end{cases}$$

Partial Eigenfunction Expansion for the Rectangle

We set

$$y_< := \min\{x_2, \xi_2\}, \qquad y_> := \max\{x_2, \xi_2\}$$

and write

$$g_m(x_2; \xi) = \frac{2}{m\pi} \frac{\sin(m\pi\xi_1/a)}{\sinh(m\pi b/a)} \sinh\left(\frac{m\pi(b - y_>)}{a}\right) \sinh\left(\frac{m\pi y_<}{a}\right)$$

Finally,

$$g(x; \xi) = \sum_{m=1}^{\infty} \frac{2}{m\pi} \frac{\sin\left(\frac{m\pi\xi_1}{a}\right) \sin\left(\frac{m\pi x_1}{a}\right) \sinh\left(\frac{m\pi(b-y_>)}{a}\right) \sinh\left(\frac{m\pi y_<}{a}\right)}{\sinh(m\pi b/a)}.$$

(Partial eigenfunction expansion)

Partial Eigenfunction Expansion for the Rectangle

We could also have expanded Green's function in terms of the x_2 eigenfunctions; this would have yielded

$$g(x;\xi) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \frac{\sin\left(\frac{n\pi\xi_2}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)\sinh\left(\frac{n\pi(a-z_>)}{b}\right)\sinh\left(\frac{n\pi z_<}{a}\right)}{\sinh(n\pi a/b)}.$$

where

$$z_< := \min\{x_1,\xi_1\}, \qquad z_> := \max\{x_1,\xi_1\}.$$

Both partial expansions give the same Green's function, as does the full eigenfunction expansion.

Which of the partial expansions is more suitable in a given situation depends on the problem.