Full Eigenfunction Expansions

Finding the Green function for the Elliptic Operator

Let $\{u_n\}$ be a basis of orthonormal eigenfunctions for (L, B).

Let g be the (unknown) Green function.

$$\int_\Omega \overline{u_n(x)} L g(x;\xi)\, dx = \overline{u_n(\xi)}$$

Since (L, B) is a self-adjoint boundary value problem,

$$\int_\Omega \overline{u_n(x)} L g(x;\xi)\, dx = \int_\Omega g(x;\xi) \overline{L u_n(x)}\, dx = \lambda_n \int_\Omega g(x;\xi) \overline{u_n(x)}\, dx.$$

It then follows that

$$\int_\Omega g(x;\xi) \overline{u_n(x)}\, dx = \frac{\overline{u_n(\xi)}}{\lambda_n}.$$

Full Eigenfunction Expansion of Green's Function

The eigenfunctions $\{u_n\}$ are an orthonormal basis of $C^2(\Omega)$.

Green's function can be expanded in a series

$$\begin{aligned} g(x;\xi) &= \sum_n \langle u_n, g(\,\cdot\,;\xi)\rangle_{L^2} u_n(x) \\ &= \sum_n \frac{u_n(x)\overline{u_n(\xi)}}{\lambda_n}. \end{aligned}$$

(Full eigenfunction expansion of g)

Example: Dirichlet Problem on a Rectangle

Let

$$L = -\Delta, \qquad \Omega = [0, a] \times [0, b] \subset \mathbb{R}^2, \qquad Bu = u|_{\partial\Omega}$$

Orthonormalized eigenfunctions are

$$u_{m,n}(x_1, x_2) = \frac{2}{\sqrt{ab}} \sin\left(\frac{m\pi x_1}{a}\right) \sin\left(\frac{n\pi x_2}{b}\right), \quad m, n = 1, 2, 3, \ldots$$

Hence,

$$g(x;\xi) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4}{ab} \frac{\sin\left(\frac{m\pi x_1}{a}\right) \sin\left(\frac{n\pi x_2}{b}\right) \sin\left(\frac{m\pi \xi_1}{a}\right) \sin\left(\frac{n\pi \xi_2}{b}\right)}{m^2\pi^2/a^2 + n^2\pi^2/b^2}$$

Example: Dirichlet Problem on a Disk

Let

$$L = -\Delta, \qquad \Omega = \{x \in \mathbb{R}^2 \colon |x| \le 1\}, \qquad Bu = u|_{\partial\Omega}$$

Orthonormalized eigenfunctions are

$$u_{m,n}(r,\varphi) = \frac{e^{in\varphi}}{\sqrt{\pi} J_n'(\alpha_{n,m})} J_n(\alpha_{n,m} r), \quad m = 1,2,3,\dots, \quad n \in \mathbb{Z}, \dots$$

where $\alpha_{n,m}$ is the mth positive zero of the nth Bessel function of the first kind, J_n. Hence,

$$g(r,\varphi;\varrho,\theta) = \frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{e^{in(\varphi-\theta)} J_n(\alpha_{n,m} r) J_n(\alpha_{n,m}\varrho)}{J_n'(\alpha_{n,m})^2 \alpha_{n,m}^2}$$