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The Eigenvalue problem for the Elliptic Operator

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$$Lu := -\operatorname{div}\big(p(x)\operatorname{grad} u\big) + q(x)u, \qquad x \in \Omega \subset \mathbb{R}^n,$$

with boundary values

$$Bu := \alpha(x)u + \beta(x)\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = \gamma(x,t).$$

We set

$$M := \big\{u \in C^2(\Omega) \colon Bu = 0\big\}.$$

The Eigenvalue Problem for the Elliptic Operator

The eigenvalue problem for L is

$$Lu = \lambda u, \qquad u \in M,$$

complex eigenvalues

Scalar product:

$$\langle u, v\rangle_{L^2} := \int_\Omega \overline{u(x)} v(x)\, dx, \qquad u, v \in C^2(\Omega)$$

complex conjugate of $u(x)$

(L, B) is a self-adjoint boundary value problem, so

$$\langle v, Lu\rangle_{L^2} = \langle Lv, u\rangle_{L^2}, \qquad \text{for } u, v \in M.$$

Eigenvalues and Eigenfunctions

This implies:

- eigenvalues λ are real numbers
- eigenfunctions u to different eigenvalues are orthogonal

A high-powered theorem (from theory of compact operators) gives:

- Eigenvalues exist.
- The eigenvalues form an infinite sequence $\lambda_1, \lambda_2, \lambda_3, \dots$ with

$$\lambda_1 \leq \lambda_2 \leq \dots \qquad \text{and} \qquad \lambda_n \xrightarrow{n\to\infty} \infty.$$

- The eigenfunctions $\{u_n\}$ give an orthonormal basis of $C^2(\Omega)$.

Positivity of Eigenvalues

We will prove that the eigenvalues can not be strictly negative:

$$\begin{aligned}
\lambda\langle u, u\rangle_{L^2} &= \langle u, Lu\rangle_{L^2} = \int_\Omega \overline{u(x)}(Lu)(x)\,dx \\
&= \int_\Omega \overline{u(x)}\big[-\operatorname{div}\big(p(x)\operatorname{grad} u(x)\big) + q(x)u(x)\big]\,dx \\
&= -\int_\Omega \big[\operatorname{div}\big(\overline{u(x)}p(x)\operatorname{grad} u(x)\big) - p(x)|\operatorname{grad} u(x)|^2\big]\,dx \\
&\quad + \int_\Omega q(x)|u(x)|^2\,dx \\
&= \int_\Omega \left(p(x)|\operatorname{grad} u(x)|^2 + q(x)|u(x)|^2\right)\,dx - \int_{\partial\Omega} p\bar{u}\frac{\partial u}{\partial n}\,d\sigma
\end{aligned}$$

Positivity of Eigenvalues

Setting $\partial\Omega = S_1 \cup S_2 \cup S_3$,

$$\begin{aligned} u\Big|_{S_1} &= 0, \\ \frac{\partial u}{\partial n}\Big|_{S_2} &= 0, \\ \frac{\partial u}{\partial n}\Big|_{S_3} &= -\frac{\alpha}{\beta}u \qquad\qquad \text{for } u \in M, \end{aligned}$$

Therefore,

$$\langle u, Lu\rangle_{L^2} = \int_\Omega \left(p(x)|\operatorname{grad} u(x)|^2 + q(x)|u(x)|^2\right)\,dx + \int_{S_3} p\frac{\alpha}{\beta}|u|^2\,d\sigma$$

so

$$\langle u, Lu\rangle_{L^2} \geq 0 \quad \text{and} \quad \langle u, Lu\rangle_{L^2} = 0 \quad \Leftrightarrow \quad q(x) \equiv \alpha(x) \equiv 0$$