Causal Fundamental Solution for the Parabolic Boundary Value Problem

Causal Fundamental Solutions

A fundamental solution $E(x, t; \xi, \tau)$ for a time-dependent PDE satisfies

$$\widetilde{L}E = \delta\big((x, t) - (\xi, \tau)\big), \qquad x, \xi \in \Omega, \quad t, \tau \in \mathbb{R}$$

E is said to be causal if

$$E(x, t; \xi, \tau) = 0 \qquad \text{whenever } t < \tau.$$

Causal Fundamental Solution

The direct Green function $g(x, t; \xi, \tau)$ for the parabolic problem in a bounded cylinder satisfies

$$\begin{aligned} \widetilde{L}g &= \delta\big((x,t)-(\xi,\tau)\big), \qquad x,\xi \in \Omega, \quad t,\tau \in (0,T). \\ Bg &= 0, \\ \widetilde{B}_1 g &= 0. \end{aligned}$$

A causal fundamental solution E already satisfies the first and the third equation!

Just as for ODEs, causal fundamental solutions may be constructed by solving PDEs with no time singularity.

Causal Fundamental Solution for an ODE

A causal fundamental solution for a first-order initial value problem:

$$\begin{aligned} a_1(t)E'(t;\tau) + a_0(t)E(t;\tau) &= \delta(t-\tau), && t,\tau \in \mathbb{R}, \\ E(t;\tau) &= 0 && t < \tau \end{aligned}$$

can be found by solving

$$\begin{aligned} a_1(t)u'(t;\tau) + a_0(t)u(t;\tau) &= 0, && t,\tau \in \mathbb{R}, \\ u(\tau;\tau) &= \frac{1}{a_1(\tau)} \end{aligned}$$

and setting

$$E(t;\tau) = H(t-\tau)u(t;\tau)$$

Analogy of the Parabolic Problem

A causal fundamental solution for the parabolic problem:

$$\begin{aligned} \varrho(x)\frac{\partial}{\partial t}E(x,t;\xi,\tau) + LE(x,t;\xi,\tau) &= \delta(x-\xi)\delta(t-\tau), && t,\tau \in \mathbb{R}, \\ E(x,t;\xi,\tau) &= 0 && t < \tau \end{aligned}$$

can be found by solving

$$\begin{aligned} \varrho(x)\frac{\partial}{\partial t}u(x,t;\xi,\tau) + Lu(x,t;\xi,\tau) &= 0, && t,\tau \in \mathbb{R}, \\ u(x,\tau;\xi,\tau) &= \frac{1}{\varrho(x)}\delta(x-\xi) \end{aligned}$$

and setting

$$E(x,t;\xi,\tau) = H(t-\tau)u(x,t;\xi,\tau)$$

Example: Heat Equation

$$p(x,t) := (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$$

solves

$$\frac{\partial u}{\partial t} - \Delta u = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \tag{I.1}$$

with initial condition

$$u(x,0) = \delta(x)$$

Hence,

$$E(x,t;\xi,\tau) = H(t-\tau)p(x-\xi,t-\tau)$$