Solution Formula for the Parabolic Boundary Value Problem

Adjoint Boundary Conditions

As usual,

$$M = \{u \in C^2(V)\colon Bu = \widetilde{B}_1 u = 0\},$$
$$M^* = \Big\{v \in C^2(V)\colon \int_{\partial V} J(u,v)\,d\vec{\sigma} = 0 \text{ for all } u \in M\Big\}.$$

Since

$$\int_{\partial V} J(u,v)\,d\vec{\sigma} = \int_{\partial\Omega}\int_0^T p(u \operatorname{grad} v - v \operatorname{grad} u)\,dt\,d\vec{\sigma} + \int_\Omega \varrho(x)\big(u(x,T)v(x,T) - u(x,0)v(x,0)\big)\,dx$$

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We see that

$$M^* = \left\{ v \in C^2(V) \colon B^* v = \widetilde{B}_1^* v = 0 \right\}$$

where

$$B^* v = Bv, \qquad \widetilde{B}_1^* v = v(x,T) = v|_{\partial V_{\text{top}}}$$

(Adjoint Boundary Conditions)

Adjoint Green Function

The adjoint Green function satisfies

$$\widetilde{L}^* g^*(x,t;\xi,\tau) = \delta\big((x,t)-(\xi,\tau)\big),$$

for

$$x,\xi \in \Omega, \qquad\qquad t,\tau \in (0,T),$$

with boundary conditions

$$\begin{aligned} Bg^* &= 0, \\ \widetilde{B}_1^* g^* &= g^*(x,T;\xi,\tau) = 0. \end{aligned}$$

Solution Formula

Start from Green's formula,

$$\int_V (v\widetilde{L}u - u\widetilde{L}^* v)\, d(x,t) = \int_{\partial\Omega}\int_0^T p(u \operatorname{grad} v - v \operatorname{grad} u)\, dt\, d\vec{\sigma} + \int_\Omega \varrho(x)\big(u(x,T)v(x,T) - u(x,0)v(x,0)\big)\, dx$$

Suppose

$$\widetilde{L}u = \varrho F(x,t), \qquad u(x,0) = f(x), \qquad Bu = \gamma(x,t)$$

and $v = g^*$ satisfies

$$\widetilde{L}^* g^* = \delta\big((x,t) - (\xi,\tau)\big), \quad g^*(x,T;\xi,\tau) = 0, \quad Bg^* = 0$$

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Solution Formula

Then

$$u(\xi,\tau) = \int_V \varrho(x)F(x,t)g^*(x,t;\xi,\tau)\,d(x,t)$$
$$+ \int_\Omega \varrho(x)g^*(x,0;\xi,\tau)f(x)\,dx$$
$$- \int_{\widetilde{S}_1} \frac{p}{\alpha}\gamma\frac{\partial g^*(\,\cdot\,;\xi,\tau)}{\partial n_x}\,d\sigma + \int_{\widetilde{S}_2\cup\widetilde{S}_3} \frac{p}{\beta}\gamma g^*(\,\cdot\,;\xi,\tau)\,d\sigma$$

where

$$\widetilde{S}_k = S_k \times [0,T] \subset \partial V_{\text{mantle}}, \qquad k = 1,2,3.$$