The Parabolic Boundary Value Problem

The Parabolic Boundary Value Problem

Recall

$$L = -\operatorname{div}(p(x)\operatorname{grad}) + q(x), \qquad x \in \Omega \subset \mathbb{R}^n,$$

and

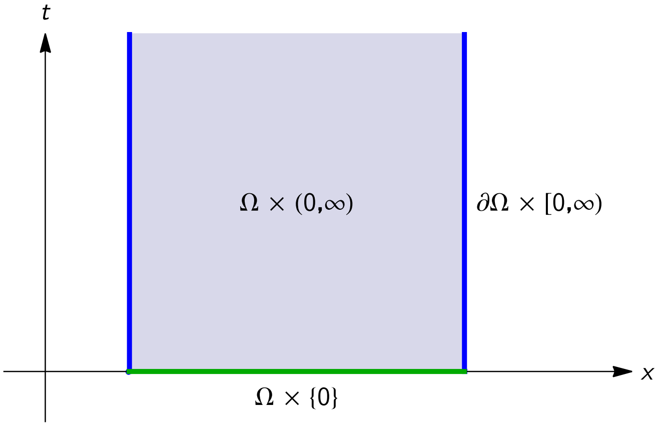
$$\widetilde{L} = \varrho(x)\frac{\partial}{\partial t} + L, \qquad (x,t) \in \Omega \times (0,\infty)$$

$\widetilde{L}$ is not self-adjoint:

$$\widetilde{L}^* = -\varrho(x)\frac{\partial}{\partial t} + L$$

Domain and Boundary of the Parabolic Problem

$$\partial(\Omega \times (0,\infty)) = \big(\Omega \times \{0\}\big) \cup \big(\partial\Omega \times [0,\infty)\big)$$



Boundary Conditions

Let $\alpha, \beta \colon \partial\Omega \to \mathbb{C}$.

$$Bu := \alpha \cdot u\big|_{\partial\Omega\times[0,\infty)} + \beta \cdot \frac{\partial u}{\partial n}\bigg|_{\partial\Omega\times[0,\infty)},$$

$$\widetilde{B}_1 u := u\big|_{\Omega\times\{0\}}.$$

We impose

$$Bu = \gamma(x,t), \qquad (x,t) \in \partial\Omega \times (0,\infty),$$

(Boundary Condition)

$$\widetilde{B}_1 u = u(x,0) = f(x) \qquad x \in \Omega$$

(Initial Condition)

Green's Formula and the Conjunct

Recall Lagrange's identity for L:

$$vLu - uLv = \operatorname{div}_x(pu \operatorname{grad}_x v - pv \operatorname{grad}_x u)$$

where div_x and grad_x emphasize the variables of differentiation.

Let $V \subset \mathbb{R}^{n+1}$ be a bounded domain. Then

$$\begin{aligned}
&\int_V (v\widetilde{L}u - u\widetilde{L}^* v)\, d(x,t) \\
&= \int_V \Big(\operatorname{div}_x(pu \operatorname{grad}_x v - pv \operatorname{grad}_x u) + \frac{d}{dt}(\varrho uv)\Big)\, d(x,t) \\
&= \int_V \operatorname{div}_{(x,t)} \begin{pmatrix} p(u \operatorname{grad}_x v - v \operatorname{grad}_x u) \\ \varrho uv \end{pmatrix} d(x,t)
\end{aligned}$$

Green's Formula and the Conjunct

Using the divergence theorem in $\mathbb{R}^{n+1}$,

$$\int_V (v\widetilde{L}u - u\widetilde{L}^* v)\, d(x,t) = \int_{\partial V} \begin{pmatrix} p(u \operatorname{grad}_x v - v \operatorname{grad}_x u) \\ \varrho uv \end{pmatrix} d\vec{\sigma}$$

so

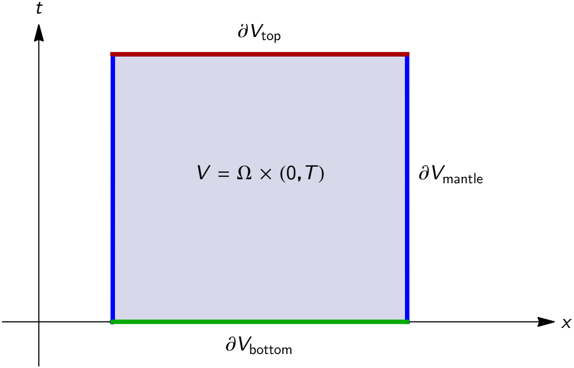
$$J(u,v) = \begin{pmatrix} p(u \operatorname{grad}_x v - v \operatorname{grad}_x u) \\ \varrho uv \end{pmatrix}.$$

is the conjunct for $\widetilde{L}$.

Restriction to a Bounded cylinder

Fix $T > 0$ and restrict the PDE to the cylinder

$$V = \Omega \times (0, T) \subset \mathbb{R}^{n+1}$$



Boundary Conditions on the Bounded Cylinder

Then

$$\begin{aligned}\partial V &= \underbrace{\Omega \times \{0\}}_{\text{“bottom”}} \cup \underbrace{\partial\Omega \times [0, T]}_{\text{“mantle”}} \cup \underbrace{\Omega \times \{T\}}_{\text{“top”}} \\ &= \partial V_{\text{bottom}} \cup \partial V_{\text{mantle}} \cup \partial V_{\text{top}}\end{aligned}$$

We have

- boundary conditions on $\partial V_{\text{mantle}}$
- initial conditions on $\partial V_{\text{bottom}}$
- no conditions on ∂V_{top}

Green's Formula for the Bounded Cylinder

From

$$\int_{\partial V} J(u,v)\,d\vec{\sigma} = \int_{\partial V_{\text{top}}} J(u,v)\,d\vec{\sigma} + \int_{\partial V_{\text{bottom}}} J(u,v)\,d\vec{\sigma} + \int_{\partial V_{\text{mantle}}} J(u,v)\,d\vec{\sigma}$$

we find

$$\int_{\partial V} J(u,v)\,d\vec{\sigma} = \int_{\partial \Omega}\int_0^T p(u \operatorname{grad} v - v \operatorname{grad} u)\,dt\,d\vec{\sigma} + \int_{\Omega} \varrho(x)\big(u(x,T)v(x,T) - u(x,0)v(x,0)\big)\,dx$$

Green's Formula for the Bounded Cylinder

$$
\begin{aligned}
&\int_V (v\widetilde{L}u - u\widetilde{L}^* v)\, d(x,t) \\
&= \int_{\partial\Omega} \int_0^T p(u \operatorname{grad} v - v \operatorname{grad} u)\, dt\, d\vec{\sigma} \\
&\qquad + \int_\Omega \varrho(x)\big(u(x,T)v(x,T) - u(x,0)v(x,0)\big)\, dx
\end{aligned}
$$