The Elliptic Boundary Value Problem

The Formal Adjoint

Definition. Let L be a partial differential operator on $\mathbb{R}^n$. The operator L^* such that

$$(LT)(\varphi) = T(L^*\varphi)$$

for any $T \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is called the formal adjoint of L.

If $L = L^*$, we say that L is formally self-adjoint.

Example.

$$L = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(p(x)\frac{\partial}{\partial x_i}\right) + q(x)$$

is formally self-adjoint.

Lagrange's Identity

For $u, v \in C^2(\mathbb{R}^n)$,

$$\begin{aligned} vLu &= -v\operatorname{div}(p\operatorname{grad} u) \\ &= -\operatorname{div}(pv\operatorname{grad} u) + p\langle \operatorname{grad} v, \operatorname{grad} u\rangle, \\ uLv &= -\operatorname{div}(pu\operatorname{grad} v) + p\langle \operatorname{grad} u, \operatorname{grad} v\rangle \end{aligned}$$

so that

$$vLu - uLv = \operatorname{div}(pu\operatorname{grad} v) - \operatorname{div}(pv\operatorname{grad} u)$$

(Lagrange's Identity)

Green's Formula and the Conjunct

Using the divergence theorem,

$$\begin{aligned}\int_\Omega (vLu - uLv)\,dx &= \int_\Omega \operatorname{div}(pu\,\operatorname{grad} v - pv\,\operatorname{grad} u)\,dx \\ &= \int_{\partial\Omega} p(u\,\operatorname{grad} v - v\,\operatorname{grad} u)\,d\vec{\sigma} \\ &= \int_{\partial\Omega} p\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\,d\sigma\end{aligned}$$

(Green's Formula)

The conjunct of L is

$$J(u, v) = p(v\,\operatorname{grad} u - u\,\operatorname{grad} v)$$

Adjoint Boundary Value Problems

Definition. Let (L, B) be a boundary value problem. Set

$$M := \{u \in C^2(\Omega) \cap C(\overline{\Omega}) \colon Bu = 0\},$$
$$M^* := \Big\{v \in C^2(\Omega) \cap C(\overline{\Omega}) \colon \int_{\partial\Omega} J(u, v)\, d\vec{\sigma} = 0 \quad \text{for all } u \in M.\Big\}$$

A boundary operator B^* such that

$$M^* = \{v \in C^2(\Omega) \colon B^* v = 0\}$$

is said to be the adjoint operator to B.

(L^*, B^*) is the adjoint boundary value problem to (L, B).

If $L = L^*$ and $M = M^*$, (L, B) is said to be self-adjoint.

Self-Adjointness

Example. We show that $M = M^*$ for

$$L = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(p(x)\frac{\partial}{\partial x_i}\right) + q(x).$$

Suppose that $u, v \in M$. Then

$$\alpha(x)u + \beta(x)\frac{\partial u}{\partial n}\bigg|_{\partial\Omega} = 0, \qquad \alpha(x)v + \beta(x)\frac{\partial v}{\partial n}\bigg|_{\partial\Omega} = 0.$$

Fix $x \in \partial\Omega$ and regard $\alpha(x), \beta(x)$ as solutions of the system

$$\begin{pmatrix} u & \frac{\partial u}{\partial n} \\ v & \frac{\partial v}{\partial n} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Self-Adjointness

This implies

$$0 = \det \begin{pmatrix} u & \frac{\partial u}{\partial n} \\ v & \frac{\partial v}{\partial n} \end{pmatrix} = u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}$$

on $\partial\Omega$.

Hence, if $u, v \in M$, then

$$\int_{\partial\Omega} J(u,v)\, d\vec{\sigma} = \int_{\partial\Omega} p\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) d\sigma = 0.$$

This shows

$$v \in M \quad \Rightarrow \quad v \in M^*$$

The converse can be shown by considering S_1, S_2 and S_3 separately.

Direct and Adjoint Green Functions

Definition. The (direct) Green function $g(x,\xi)$ for (L,B) satisfies

$$Lg = \delta(x-\xi), \qquad Bg = 0.$$

while the adjoint Green function g^* satisfies

$$L^*g^* = \delta(x-\xi), \qquad B^*g^* = 0.$$

Solution Formula for the Elliptic Problem

Suppose u solves (L, B) with data $(\varrho F, \gamma)$ and $g^* = g$ is the Green function for $(L^*, B^*) = (L, B)$.

Then Green's formula gives

$$\begin{aligned} u(\xi) &= \int_\Omega g(x,\xi)\varrho(x)F(x)\,dx - \int_{\partial\Omega} p\left(u\frac{\partial g}{\partial n} - g\frac{\partial u}{\partial n}\right)\,d\sigma \\ &= \int_\Omega g(x,\xi)\varrho(x)F(x)\,dx \\ &\qquad - \int_{S_1} \frac{p}{\alpha}\gamma\frac{\partial g(\,\cdot\,,\xi)}{\partial n}\,d\sigma + \int_{S_2\cup S_3} \frac{p}{\beta}\gamma g(\,\cdot\,,\xi)\,d\sigma \end{aligned}$$