Solution Formula via Modified Green Functions

Modified Adjoint Green function

The modified adjoint Green g_M^* satisfies

$$
\begin{aligned}
L^* g_M^*(x,\xi) &= \delta(x-\xi) - \sum_{i=1}^{k} u^{(i)}(\xi) u^{(i)}(x), \\
B_1^* g_M^* &= 0 \\
&\vdots \\
B_p^* g_M^* &= 0.
\end{aligned}
$$

where $u^{(1)}, \ldots, u^{(k)}$ are the k orthonormalized non-trivial solutions of the completely homogeneous direct problem.

A Solution Formula

Suppose that u is a solution of

$$Lu = f, \qquad B_1 u = \cdots = B_p u = 0.$$

Then

$$\begin{aligned} 0 &= J(u, g_M^*)\Big|_a^b \\ &= \int_a^b \big(g_M^*(x,\xi) Lu(x) - u(x) L^* g_M^*(x,\xi)\big)\, dx \\ &= \int_a^b g_M^*(x,\xi) f(x) - u(x)\delta(x-\xi) + u(x) \sum_{i=1}^k u^{(i)}(\xi) u^{(i)}(x)\, dx \\ &= -u(\xi) + \int_a^b g_M^*(x,\xi) f(x)\, dx + \sum_{i=1}^k \langle u, u^{(i)}\rangle u^{(i)}(\xi). \end{aligned}$$

A Solution Formula

This implies

$$u(x) = \int_a^b g_M^*(\xi, x) f(\xi)\, d\xi + \sum_{i=1}^{k} \langle u, u^{(i)} \rangle u^{(i)}(x)$$

Since we can always add solutions to the completely homogeneous problem to u, we can take

$$u(x) = \int_a^b g_M^*(\xi, x) f(\xi)\, d\xi$$

Express the solution formula in terms of the modified direct Green function.

The Modified Direct and Adjoint Green's functions

By Green's formula,

$$
\begin{aligned}
0 &= J(g_M, g_M^*)\Big|_a^b \\
&= \int_a^b g_M^*(x,\eta) L g_M(x,\xi) - g_M(x,\xi) L^* g_M^*(x,\eta)\,dx \\
&= g_M^*(\xi,\eta) - g_M(\eta,\xi) - \sum_{i=1}^{k} v^{(i)}(\xi) \underbrace{\int_a^b v^{(i)}(x) g_M^*(x,\eta)\,dx}_{=\langle v^{(i)}, g_M^*(\,\cdot\,,\eta)\rangle} \\
&\qquad + \sum_{i=1}^{k} u^{(i)}(\eta) \underbrace{\int_a^b u^{(i)}(x) g_M(x,\xi)\,dx}_{=\langle u^{(i)}, g_M(\,\cdot\,,\xi)\rangle}
\end{aligned}
$$

The Modified Direct and Adjoint Green's functions

$$g_M^*(\xi, x) = g_M(x, \xi) + \sum_{i=1}^{k} \big(v^{(i)}(\xi) \langle v^{(i)}, g_M^*(\,\cdot\,, x) \rangle - u^{(i)}(x) \langle u^{(i)}, g_M(\,\cdot\,, \xi) \rangle \big)$$

Then

$$u(x) = \int_a^b g_M^*(\xi, x) f(\xi)\, d\xi + \sum_{i=1}^{k} \langle u, u^{(i)} \rangle u^{(i)}(x)$$

becomes

$$u(x) = \int_a^b g_M(x, \xi) f(\xi)\, d\xi - \sum_{i=1}^{k} u^{(i)}(x) \int_a^b \langle u^{(i)}, g_M(\,\cdot\,, \xi) \rangle f(\xi)\, d\xi + \sum_{i=1}^{k} \langle u, u^{(i)} \rangle u^{(i)}(x)$$

A Solution Formula

Write the last equation as

$$u(x) - \int_a^b g_M(x,\xi) f(\xi)\, d\xi = \sum_{i=1}^{k} \left\langle u - \int_a^b g_M(\,\cdot\,,\xi) f(\xi)\, d\xi, u^{(i)} \right\rangle u^{(i)}(x)$$

Geometrically,

$$u - \int_a^b g_M(\,\cdot\,,\xi) f(\xi)\, d\xi \;\in\; \operatorname{span}\{u^{(1)}, \dots, u^{(k)}\},$$

i.e.,

$$u(x) = \int_a^b g_M(x,\xi) f(\xi)\, d\xi + \sum_{i=1}^{k} c_i u^{(i)}(x)$$

where $c_1, \dots, c_k$ are arbitrary constants.