Modified Green Functions

Existence of Green's function

Solvability conditions for

$$Lu(x,\xi) = f, \quad a < x < b, \qquad B_1 u = \cdots = B_p u = 0$$

are

$$\int_a^b f(x) v^{(j)}(x)\, dx = 0 \qquad j = 1, \ldots, k. \tag{I.1}$$

where $v^{(1)}, \ldots, v^{(k)}$ are independent solutions of the completely homogeneous adjoint problem.

If $f(x) = \delta(x - \xi)$ these are not satisfied for all $\xi \in (a, b)$, so Green's function doesn't exist.

Example: Formally Self-Adjoint Problem

$$\begin{aligned} -u'' + u' &= \delta(x-\xi), \qquad\qquad 0 < x < 1, \\ u(1) - u(0) &= 0, \\ u'(1) - u'(0) &= 0 \end{aligned}$$

$v(x) = 1$ is a solution of the completely homogeneous adjoint problem.

Solvability condition for Green's function:

$$\int_0^1 \delta(x-\xi)\, dx = 0.$$

Not satisfied.

Approach: Derive a "modified" Green function instead.

Orthonormalization

Non-trivial solutions of the completely homogeneous adjoint problem:

$$v^{(1)}, \dots, v^{(k)}$$

Orthonormalize with respect to

$$\langle f, g \rangle_{L^2([a,b])} := \int_a^b f(x) g(x)\, dx.$$

so that

$$\int_a^b v^{(i)}(x) v^{(j)}(x)\, dx = \delta_{ij} = \begin{cases} 0 & i \neq j, \\ 1 & i = j. \end{cases}$$

Modified Equation for Green's Function

Instead of $Lu = \delta(x - \xi)$ solve

$$\begin{aligned} Lu &= \delta(x - \xi) - v^{(1)}(\xi)v^{(1)}(x) - \cdots - v^{(k)}(\xi)v^{(k)}(x) \\ &=: f(x) \end{aligned}$$

Solvability conditions for $j = 1, \ldots, k$ are satisfied:

$$\begin{aligned} \int_a^b f(x)v^{(j)}(x)\,dx &= \int_a^b \Big(\delta(x - \xi) - \sum_{i=1}^k v^{(i)}(\xi)v^{(i)}(x)\Big)v^{(j)}(x)\,dx \\ &= v^{(j)}(\xi) - \sum_{i=1}^k v^{(i)}(\xi) \underbrace{\int_a^b v^{(i)}(x)v^{(j)}(x)\,dx}_{=\delta_{ij}} \\ &= 0. \end{aligned}$$

Modified Green function

Definition. The modified (direct) Green function is defined by

$$Lg_M(x,\xi) = \delta(x-\xi) - \sum_{i=1}^{k} v^{(i)}(\xi)v^{(i)}(x),$$

$$B_1 g_M = 0$$

$$\vdots$$

$$B_p g_M = 0.$$

where $v^{(1)}, \dots, v^{(k)}$ are the k orthonormalized non-trivial solutions of the completely homogeneous adjoint problem.

Constructing the Modified Green Function

Suppose that $v^{(1)}, \dots, v^{(k)}$ are the k non-trivial, orthonormalized solutions of the adjoint problem.

(i) Find a fundamental solution $E(x, \xi)$ such that

$$LE = \delta(x - \xi)$$

(ii) Find k solutions

$$w^{(1)}, \dots, w^{(k)}$$

of the inhomogeneous equations

$$Lw^{(i)} = v^{(i)}, \qquad i = 1, \dots, k,$$

(without regard to boundary conditions). Then

$$L\Big(E(x,\xi) - \sum_{i=1}^{k} v^{(i)}(\xi) w^{(i)}(x)\Big) = \delta(x - \xi) - \sum_{i=1}^{k} v^{(i)}(\xi) v^{(i)}(x)$$

Constructing the Modified Green Function

(iii) Find p independent solutions of the homogeneous equation $Lu = 0$ and add them to

$$E(x,\xi) - \sum_{i=1}^{k} v^{(i)}(\xi) w^{(i)}(x)$$

in order to satisfy the boundary conditions

$$B_1 g = \cdots = B_p g = 0.$$

Example

$$
\begin{aligned}
Lu &= u'', && 0 < x < 1, \\
B_1 u &= u(0) + u(1), \\
B_2 u &= u'(0) - u'(1)
\end{aligned}
$$

The completely homogeneous problem has a non-trivial solution,

$$u^{(1)}(x) = 1 - 2x.$$

Green's formula is

$$\int_0^1 (vu'' - uv'')\, dx = vu' - uv'|_0^1.$$

We set

$$B_3 u = u(0), \qquad B_4 u = u'(0).$$

Example

$$
\begin{aligned}
J(u,v)\big|_0^1 &= v(1)u'(1) - u(1)v'(1) - v(0)u'(0) + u(0)v'(0) \\
&= -v(1)B_2u + v(1)B_4u - v'(1)B_1u + v'(1)B_3u \\
&\quad - v(0)B_4u + v'(0)B_3u \\
&= -v'(1)B_1u - v(1)B_2u + (v'(1) + v'(0))B_3u \\
&\quad + (v(1) - v(0))B_4u
\end{aligned}
$$

The adjoint boundary conditions are

$$
\begin{aligned}
B_1^*v &= v(1) - v(0), & B_2^*v &= v'(1) + v'(0), \\
B_3^*v &= -v(1), & B_4^*v &= -v'(1).
\end{aligned}
$$

Example

The completely homogeneous adjoint problem

$$\begin{aligned} v'' &= 0, && 0 < x < 1, \\ v(0) &= v(1), \\ v'(0) &= -v'(1) \end{aligned}$$

has a non-trivial solution

$$v^{(1)}(x) = 1.$$

Hence the problem

$$Lu = f, \qquad 0 < x < 1, \qquad B_1 u = 0, \qquad B_2 u = 0$$

is solvable if and only if

$$\int_0^1 f(x)\, dx = 0.$$

Example

Green's function doesn't exist.

Construct a modified Green function:

(i) Causal fundamental solution for L:

$$E(x,\xi) = H(x-\xi)\cdot(x-\xi)$$

(ii) Find a solution of

$$Lw = v^{(1)}(x) = 1.$$

Choose

$$w(x) = \frac{x^2}{2}.$$

Example

(iii) Add solutions of the homogeneous equation $Lu = 0$,

$$g_M(x,\xi) = H(x-\xi)(x-\xi) - \frac{x^2}{2} + a + bx, \quad a, b \in \mathbb{R},$$

to satisfy

$$B_1 g_M = B_2 g_M = 0.$$

In particular,

$$0 = g_M(1,\xi) + g_M(0,\xi) = (1-\xi) - \frac{1}{2} + a + b + a$$

so $2a + b = 1/2 - \xi$ and

$$0 = g_M'(1,\xi) - g_M'(0,\xi) = 1 - 1 + b - b$$

so b is arbitrary.

Construction of the Modified Green's Function

Set $b = 0$ and obtain

$$g_M(x,\xi) = H(x-\xi)(x-\xi) - \frac{x^2-\xi}{2} - \frac{1}{4}.$$