Solvability Conditions

Existence and Uniqueness

If the boundary value problem

$$(L, B_1, \dots, B_p)$$

with data

$$(0; 0, \dots, 0)$$

has only the trivial solution,

- Green's function can be constructed and
- there exists a solution formula for any data $\{f; \gamma_1, \dots, \gamma_p\}$.

(Existence and uniqueness of the solution for the general problem)

The Fredholm Alternative

Fredholm Alternative.

- Either the completely homogeneous problem has a non-trivial solution,
- Or the solution to the problem with data $\{f; 0, \dots, 0\}$ exists and is unique.

Relationship to the Adjoint Problem

The completely homogeneous direct problem is

$$Lu = 0, \qquad x \in (a, b), \qquad B_1 u = \cdots = B_p u = 0. \qquad (*)$$

There is a relationship to the adjoint problem

$$L^* v = 0, \qquad x \in (a, b), \qquad B_1^* v = \cdots = B_p^* v = 0 \qquad (**)$$

as follows:

- If $(*)$ has only the trivial solution, then $(**)$ also has only the trivial solution.
- If there are k independent, non-trivial solutions $u^{(1)}, \ldots, u^{(k)}$ of $(*)$, then $(**)$ also has k independent, non-trivial solutions $v^{(1)}, \ldots, v^{(k)}$.

Solvability via the Adjoint Problem

Consider now the problem

$$Lu = f, \qquad x \in (a, b), \qquad B_1 u = \cdots = B_p u = 0.$$

Suppose there exists a solution u and let v be any non-trivial solution of the completely homogeneous adjoint problem $(L^*, B_1^*, \ldots, B_p^*)$. Then

$$\begin{aligned}\int_a^b f(x)v(x)\,dx &= \int_a^b \big(v(x)Lu(x) - u(x)L^* v(x)\big)\,dx \\ &= J(u, v)\big|_a^b \\ &= 0\end{aligned}$$

Solvability via the Adjoint Problem

Hence, if $v^{(1)}, \dots, v^{(k)}$ are k independent, non-trivial solutions of the completely homogeneously adjoint problem $(L^*, B_1^*, \dots, B_p^*)$, a necessary condition for the solvability of

$$(L, B_1, \dots, B_p) \qquad \text{with data} \qquad (f; 0, \dots, 0)$$

is

$$\int_a^b f(x)v^{(1)}(x)\,dx = \cdots = \int_a^b f(x)v^{(k)}(x)\,dx = 0.$$

It can be shown that this condition is also sufficient, i.e., a solution exists if and only if f satisfies these k equations.

Example

$$
\begin{aligned}
-u'' + u' &= f, & 0 < x < 1,\\
u(1) - u(0) &= 0,\\
u'(1) - u'(0) &= 0.
\end{aligned}
$$

The fully homogeneous adjoint problem is

$$
\begin{aligned}
-v'' - v' &= 0, & 0 < x < 1,\\
v(1) - v(0) &= 0,\\
v'(1) - v'(0) &= 0.
\end{aligned}
$$

which has non-trivial solution $v(x) = c$, $c \in \mathbb{R}$.

Hence, a solution will exist if and only if

$$\int_0^1 f(x)\,dx = 0.$$

Example

$$u' + u = f, \qquad 0 < x < 1, \qquad u(0) - e \cdot u(1) = 0.$$

The adjoint homogeneous problem is

$$-v' + v = 0, \qquad 0 < x < 1, \qquad -e \cdot v(0) + v(1) = 0.$$

which has non-trivial solution

$$v(x) = c \cdot e^x, \qquad c \in \mathbb{R}.$$

The (necessary and sufficient) solvability condition is

$$\int_0^1 f(x)e^x \, dx = 0.$$

Solvability of the General Inhomogeneous Problem

Consider now the problem

$$Lu = f, \qquad x \in (a,b), \qquad B_1 u = \gamma_1, \qquad \dots, \qquad B_p u = \gamma_p.$$

Suppose u is a solution and v any non-trivial solution of the completely homogeneous adjoint problem $(L^*, B_1^*, \dots, B_p^*)$. Then

$$\begin{aligned}\int_a^b f(x)v(x)\,dx &= \int_a^b \big(v(x)Lu(x) - u(x)L^*v(x)\big)\,dx \\ &= J(u,v)\big|_a^b \\ &= \gamma_1 B_{2p}^* v + \cdots + \gamma_p B_{p+1}^* v\end{aligned}$$

where $B_{p+1}^*, \dots, B_{2p}^*$ are the additional adjoint boundary functionals introduced previously.

Solvability of the General Inhomogeneous Problem

If $v^{(1)}, \ldots, v^{(k)}$ are k non-trivial solution of the completely homogeneous adjoint problem $(L^*, B_1^*, \ldots, B_p^*)$, the solvability conditions are

$$\int_a^b f(x) v^{(1)}(x)\, dx = \gamma_1 B_{2p}^* v^{(1)} + \cdots + \gamma_p B_{p+1}^* v^{(1)},$$

$$\int_a^b f(x) v^{(2)}(x)\, dx = \gamma_1 B_{2p}^* v^{(2)} + \cdots + \gamma_p B_{p+1}^* v^{(2)},$$

$$\vdots$$

$$\int_a^b f(x) v^{(k)}(x)\, dx = \gamma_1 B_{2p}^* v^{(k)} + \cdots + \gamma_p B_{p+1}^* v^{(k)}.$$

Example

$$u' + u = f, \qquad 0 < x < 1, \qquad u(0) - e \cdot u(1) = \gamma_1.$$

We have $L^* v = -v' + v$ and

$$\begin{aligned} J(u,v)\Big|_0^1 &= u(1)v(1) - u(0)v(0) \\ &= \underbrace{\big(u(0) - e \cdot u(1)\big)}_{=B_1 u} \underbrace{-v(0)}_{=B_2^* v} + \underbrace{u(1)}_{=B_2 u} \underbrace{\big(v(1) - ev(0)\big)}_{=B_1^* v} \end{aligned}$$

We have already seen that

$$v(x) = c \cdot e^x$$

solves the fully homogeneous adjoint problem (L^*, B_1^*).

Example

Then

$$J(u,v)\Big|_0^1 = \gamma_1 B_2^* v = -\gamma_1 \cdot c.$$

A solution to (L, B_1) with data (f, γ_1) exists if and only if

$$\int_0^1 c \cdot e^x f(x)\,dx = -c\gamma_1$$

or

$$\int_0^1 e^x f(x)\,dx = -\gamma_1.$$