Boundary Value Problems of General Order

Boundary Value Problems of Order p

Consider the problem $(L, B_1, \dots, B_p)$ on $[a, b] \subset \mathbb{R}$, where

$$L = a_p \frac{d^p}{dx^p} + a_{p-1} \frac{d^{p-1}}{dx^{p-1}} + \cdots + a_1 \frac{d}{dx} + a_0.$$

with $a_0, \dots, a_p \in C([a, b])$ and $a_p(x) \neq 0$ for all $x \in [a, b]$.

We have boundary functionals

$$B_1 u := \sum_{k=1}^{p} \alpha_{1k} u^{(k-1)}(a) + \sum_{k=1}^{p} \beta_{1k} u^{(k-1)}(b),$$

$$\vdots$$

$$B_p u := \sum_{k=1}^{p} \alpha_{pk} u^{(k-1)}(a) + \sum_{k=1}^{p} \beta_{pk} u^{(k-1)}(b).$$

Boundary Value Problems of Order p

Assumptions:

(i) The row vectors

$$(\alpha_{i1}, \dots, \alpha_{ip}, \beta_{i1}, \dots, \beta_{ip})$$

are independent.

(ii) The completely homogeneous problem has only the trivial solution.

We seek to solve the problem $(L, B_1, \dots, B_p)$ for data

$$\{f; \gamma_1, \dots, \gamma_p\}.$$

Boundary Value Problems of Order p

We define

$$M := \{u \in C^p(a,b) \colon B_1 u = \cdots = B_p u = 0\},$$
$$M^* := \{v \in C^p(a,b) \colon J(u,v)\big|_a^b = 0 \text{ for all } u \in M\}.$$

The boundary value problem $(L, B_1, \ldots, B_p)$ is said to be self-adjoint if

$$L = L^* \qquad \text{and} \qquad M = M^*.$$

Goal: characterize M^* through adjoint boundary functionals

$$B_1^*, \ldots, B_p^*$$

The Conjunct

Recall that

$$J(u,v) = \sum_{k=1}^{p} \sum_{i+j=k-1} (-1)^i D^i(a_m v) D^j u.$$

We express $J(u,v)\big|_a^b$ in the form

$$J(u,v)\big|_a^b = \sum_{k=1}^{p} (A_{2p+1-k} v) u^{(k-1)}(a) + \sum_{k=1}^{p} (A_{p+1-k} v) u^{(k-1)}(b)$$

with boundary functionals A_k, $k = 1, \dots, 2p$.

The right-hand side is a linear combination of the $2p$ terms

$$u(a), \dots, u^{(p-1)}(a), \qquad u(b), \dots, u^{(p-1)}(b).$$

Additional Boundary Functionals

We now define p additional boundary functionals as follows:

$$B_{p+1}u := \sum_{k=1}^{p} \alpha_{(p+1)k} u^{(k-1)}(a) + \sum_{k=1}^{p} \beta_{(p+1)k} u^{(k-1)}(b)$$

$$\vdots$$

$$B_{2p}u := \sum_{k=1}^{p} \alpha_{(2p)k} u^{(k-1)}(a) + \sum_{k=1}^{p} \beta_{(2p)k} u^{(k-1)}(b)$$

such that all $2p$ row vectors

$$(\alpha_{i1}, \dots, \alpha_{ip}, \beta_{i1}, \dots, \beta_{ip}), \qquad i = 1, \dots, 2p$$

are independent.

Adjoint Boundary Functionals

We can then write

$$\begin{aligned} J(u,v)\Big|_a^b &= \sum_{k=1}^{2p} (B^*_{2p+1-k}v)\cdot B_k u \\ &= (B^*_{2p}v)B_1 u + \cdots + (B^*_{p+1}v)B_p u \\ &\quad + (B^*_p v)B_{p+1}u + \cdots + (B^*_1 v)B_{2p}u. \end{aligned}$$

with certain boundary functionals B^*_{2p+1-k}, $k = 1, \dots, 2p$.

If $u \in M$, $J(u,v)\Big|_a^b$ vanishes if v satisfies

$$B^*_1 v = \cdots = B^*_p v = 0,$$

so these are just the adjoint boundary functionals.

Example

$$L = \frac{d^2}{dx^2} + x^2 \frac{d}{dx} + 1 \qquad \text{on } (0,1) \subset \mathbb{R}$$

with

$$B_1 u = u(0) + u(1), \qquad B_2 u = u'(1).$$

The boundary functionals correspond to row vectors

$$(1,0,1,0) \qquad \text{and} \qquad (0,0,0,1).$$

We add two functionals, $B_3 u = u(1)$ and $B_4 u = u'(0)$, which correspond to row vectors

$$(0,0,1,0) \qquad \text{and} \qquad (0,1,0,0).$$

Example

The conjunct is

$$J(u,v) = (u'v - uv') + 2xuv.$$

and

$$J(u,v)\big|_0^1 = v'(0)u(0) - v(0)u'(0) + \big(2v(1) - v'(1)\big)u(1) + v(1)u'(1)$$

Since $u(0) = B_1 u - B_3 u$, we have

$$\begin{aligned} J(u,v)\big|_0^1 = {} & \underbrace{v'(0)}_{=:B_4^* v} B_1 u + \underbrace{v(1)}_{=:B_3^* v} \cdot B_2 u + \underbrace{\big(2v(1) - v'(1) - v'(0)\big)}_{=:B_2^* v} B_3 u \\ & + \underbrace{v(0)}_{=:B_1^* v} B_4 u \end{aligned}$$

Example

We hence obtain the adjoint boundary functionals

$$B_1^* v = v(0), \qquad B_2^* v = v'(0) + 2v(1) - v'(1)$$

Solution Formula

As in the previous section, we define the direct and adjoint Green functions to satisfy

$$Lg(x,\xi) = \delta(x-\xi), \qquad B_1 g = \cdots = B_p g = 0,$$

$$L^* g^*(x,\xi) = \delta(x-\xi), \qquad B_1^* g^* = \cdots = B_p^* g^* = 0,$$

and we can again show that

$$g^*(x,\xi) = g(\xi,x).$$

Then the solution of $(L, B_1, \ldots, B_p)$ with data $\{f; \gamma_1, \ldots, \gamma_p\}$ is

$$u(x) = \int_a^b g(x,\xi) f(\xi)\, d\xi - J(u, g(x,\,\cdot\,))\Big|_a^b.$$