Point Sources and Green Functions

A Point Source

Fundamental equation for the temperature ($\alpha^2 = 1$):

$$\int_0^1 \frac{\partial \theta}{\partial t}\,dx = \int_0^1 \frac{\partial^2 \theta}{\partial x^2}\,dx + Q(t)$$

Assumption: Point heat source located at $0 < \xi < 1$ such that

$$Q(t) = 1$$

There exists no density q such that

$$Q(t) = \int_0^1 q(x,t)\,dx$$

Differential Equation for a Point Source

Equilibrium equation, $\theta(x,t) = u(x)$:

$$-\frac{d^2u}{dx^2} = 0, \qquad 0 < x < 1,\ x \neq \xi.$$

(Differential Equation is not defined for $x = \xi$!)

Denote by $g(x,\xi)$ the solution with

$$g(0,\xi) = g(1,\xi) = 0 \qquad \text{and} \qquad Q(t) = 1.$$

(Green's Function)

The Heat Balance Equation

The differential equation implies

$$g(x,\xi) = \begin{cases} Ax & 0 < x < \xi, \\ B(1-x) & \xi < x < 1, \end{cases} \qquad A, B \in \mathbb{R}. \tag{I.1}$$

Problem: g classical $\Rightarrow A = B = 0$

The equation does not take the point source into account.

Consider instead the heat balance equation

$$\int_a^b \frac{\partial^2 u}{\partial x^2}\,dx + Q(t) = 0$$

which holds for any $a, b \in [0,1]$.

The Jump Condition

In particular, for $\varepsilon > 0$,

$$\int_{\xi-\varepsilon}^{\xi+\varepsilon} \frac{\partial^2 g}{\partial x^2}\, dx = -Q = -1$$

so

$$g'(x,\xi)|_{x=\xi+\varepsilon} - g'(x,\xi)|_{x=\xi-\varepsilon} = -1.$$

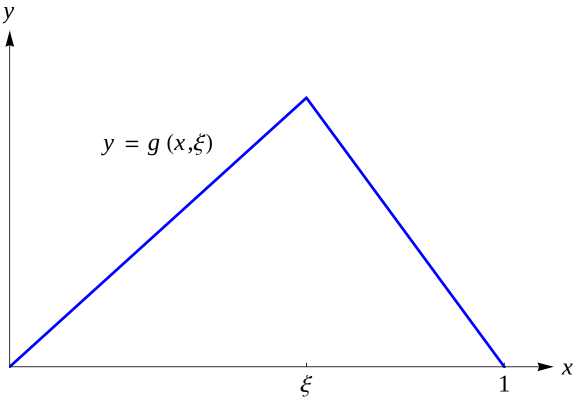
or

$$\lim_{x \nearrow \xi} g'(x,\xi) - \lim_{x \searrow \xi} g'(x,\xi) = -1$$

(Jump Condition)

The Solution for a Point Source

$$g(x,\xi) = \begin{cases} (1-\xi)x & 0 \le x < \xi, \\ (1-x)\xi & \xi \le x \le 1. \end{cases}$$



Several Point Sources

Generalization: Two point sources

- located at ξ_1 and ξ_2
- generating heat q_1 and q_2

The problem is linear, so the solution (temperature distribution) is

$$u(x) = q_1 \cdot g(x, \xi_1) + q_2 \cdot g(x, \xi_2).$$

Check:

(i) u satisfies the boundary conditions $u(0) = u(1) = 0$.

(ii) u satisfies $u''(x) = 0$ for any $x \neq \xi_1, \xi_2$.

(iii) u satisfies the heat balance on any subinterval of $[0, 1]$.