The Adjoint Second-Order Boundary Value Problem

The Formal Adjoint and Green's Formula

The formal adjoint of

$$L = a_2 \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_0$$

is

$$L^* = a_2 \frac{d^2}{dx^2} + (2a_2' - a_1) \frac{d}{dx} + (a_2'' - a_1' + a_0).$$

Green's identity is

$$\int_a^b (vLu - uL^* v) = J(u,v)\big|_a^b$$

with the conjunct

$$J(u,v) = a_2 \big(vu' - uv' \big) + \big(a_1 - a_2' \big) uv.$$

Adjoint Boundary Value Problems

We want to solve the problem (L, B_1, B_2) on $(a, b) \subset \mathbb{R}$ with

$$B_1 u = \alpha_{11} u(a) + \alpha_{12} u'(a) + \beta_{11} u(b) + \beta_{12} u'(b),$$
$$B_2 u = \alpha_{21} u(a) + \alpha_{22} u'(a) + \beta_{21} u(b) + \beta_{22} u'(b),$$

where $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$, $i, j = 1, 2$.

Suppose that u satisfies

$$B_1 u = B_2 u = 0.$$

Question: For which functions v is $J(u, v)|_a^b = 0$?

Adjoint Boundary Functionals

Definition.

$$M := \{u \in C^2(a,b) \colon B_1 u = B_2 u = 0\},$$
$$M^* := \{v \in C^2(a,b) \colon J(u,v)|_a^b = 0 \text{ for all } u \in M\}.$$

There exist so-called adjoint boundary functionals B_1^* and B_2^* such that

$$M^* = \{v \in C^2(a,b) \colon B_1^* v = B_2^* v = 0\}$$

The adjoint boundary functionals have the form

$$B_1^* u = \alpha_{11}^* u(a) + \alpha_{12}^* u'(a) + \beta_{11}^* u(b) + \beta_{12}^* u'(b),$$
$$B_2^* u = \alpha_{21}^* u(a) + \alpha_{22}^* u'(a) + \beta_{21}^* u(b) + \beta_{22}^* u'(b),$$

where $\alpha_{ij}^*, \beta_{ij}^* \in \mathbb{R}$, $i, j = 1, 2$.

Adjoint Boundary Functionals

The existence of B_1^* and B_2^* follows from

$$J(u,v)\Big|_a^b = a_2\big(vu' - uv'\big)\Big|_a^b + \big(a_1 - a_2'\big)uv\Big|_a^b$$

and then "factoring out" B_1u and B_2u in the equation

$$J(u,v)\Big|_a^b = 0.$$

While M^* is completely determined by M, B_1^*, B_2^* are not unique.

For example, we can replace B_1^*, B_2^* by

$$\widetilde{B}_1^* = B_1^* + B_2^*, \qquad \widetilde{B}_2^* = B_1^* - B_2^*$$

without affecting M^*.

Example of Adjoint Boundary Value Functionals

$$L = \frac{d^2}{dx^2} \qquad \text{on } (0,1) \subset \mathbb{R}$$

with

$$B_1 u = u'(0) - u(1), \qquad\qquad B_2 u = u'(1).$$

The conjunct is

$$\begin{aligned} J(u,v)\big|_0^1 &= vu' - uv'\big|_0^1 \\ &= v(1)u'(1) - u(1)v'(1) - v(0)u'(0) + u(0)v'(0). \end{aligned}$$

Now if $u \in M = \{u \in C^2([0,1]) \colon B_1 u = B_2 u = 0\}$, then

$$J(u,v)\big|_0^1 = -u'(0)[v'(1) + v(0)] + u(0)v'(0).$$

Example of Adjoint Boundary Value Functionals

Hence,

$$\begin{aligned} M^* &= \{v \in C^2([0,1])\colon J(u,v)|_a^b = 0 \text{ for all } u \in M\} \\ &= \{v \in C^2([0,1])\colon v'(1) + v(0) = 0 \text{ and } v'(0) = 0\} \end{aligned}$$

A possible choice of adjoint boundary functionals is

$$B_1^* v = v'(1) + v(0), \qquad\qquad B_2^* v = v'(0).$$

Adjoint Boundary Value Problems

Definition. The boundary value problem

$$(L^*, B_1^*, B_2^*)$$

is said to be the adjoint of

$$(L, B_1, B_2)$$

.

(L, B_1, B_2) is called self-adjoint if

$$L = L^* \qquad \text{and} \qquad M = M^*.$$