# Green's Function and a Solution Formula for Second-Order Boundary Value Problems

# Mixed Boundary Conditions

In the general case, we have

$$B_1 g := \alpha_{11} g(a) + \alpha_{12} g'(a) + \beta_{11} g(b) + \beta_{12} g'(b) = 0,$$
$$B_2 g := \alpha_{21} g(a) + \alpha_{22} g'(a) + \beta_{21} g(b) + \beta_{22} g'(b) = 0,$$

It is possible to find a non-trivial function $u_1$ satisfying

$$Lu_1 = 0, \qquad B_1 u_1 = 0.$$

by solving $Lu_1 = 0$ with the separated boundary conditions

$$\alpha_{11} u_1(a) + \alpha_{12} u_1'(a) = 1,$$
$$\beta_{11} u_1(b) + \beta_{12} u_1'(b) = -1.$$

Similarly, there exists a non-trivial $u_2$ such that

$$Lu_2 = 0, \qquad B_2 u_2 = 0.$$

# Green's Function for Mixed Boundary Conditions

We construct Green's function from the sum of the causal fundamental solution

$$E(x,\xi) = H(x-\xi)u_\xi(x)$$

and $u_1$ and $u_2$:

$$g(x,\xi) = H(x-\xi)u_\xi(x) + c_1 \cdot u_1(x) + c_2 \cdot u_2(x)$$

where $c_1, c_2 \in \mathbb{C}$ may depend on $\xi$.

The constants are determined through

$$\begin{aligned} B_1 g &= \beta_{11} u_\xi(b) + \beta_{12} u_\xi'(b) + c_2 \cdot B_1 u_2 &= 0, \\ B_2 g &= \beta_{21} u_\xi(b) + \beta_{22} u_\xi'(b) + c_1 \cdot B_2 u_1 &= 0. \end{aligned}$$

# Example for Mixed Boundary Conditions

$$\begin{aligned} Lu &= u'' && \text{on } (0,1) \subset \mathbb{R}, \\ B_1 u &= u(0) + u(1) \\ B_2 u &= u'(0) + u'(1) \end{aligned}$$

We first find a causal fundamental solution by solving

$$u_\xi'' = 0, \qquad u_\xi(\xi) = 0, \qquad u_\xi'(\xi) = 1.$$

This gives

$$u_\xi(x) = x - \xi$$

so the casual fundamental solution is

$$E(x, \xi) = H(x - \xi) \cdot (x - \xi).$$

# Example for Mixed Boundary Conditions

We find a non-trivial function $u_1$ such that

$$u_1'' = 0, \qquad B_1 u_1 = u_1(0) + u_1(1) = 0.$$

We take

$$u_1(x) = 1 - 2x.$$

Next we choose a function $u_2$ such that

$$u_2'' = 0, \qquad B_2 u_2 = u_2'(0) + u_2'(1) = 0.$$

and we can take

$$u_2(x) = 1.$$

# Example for Mixed Boundary Conditions

Then Green's function is

$$g(x,\xi) = H(x-\xi)\cdot(x-\xi) + c_1(1-2x) + c_2, \quad 0<\xi<1,$$

and the parameters $c_1, c_2 \in \mathbb{R}$ are determined through

$$\begin{aligned}
B_1 g &= g(0,\xi) + g(1,\xi) \\
&= c_1 + c_2 + 1 - \xi - c_1 + c_2 \\
&= 0, \\
B_2 g &= g'(0,\xi) + g'(1,\xi) \\
&= -2c_1 + 1 - 2c_1 \\
&= 0
\end{aligned}$$

which gives

$$c_1 = \frac{1}{4}, \qquad c_2 = \frac{\xi - 1}{2}.$$

# Example for Mixed Boundary Conditions

We finally have

$$g(x,\xi) = H(x-\xi)\cdot(x-\xi) - \frac{x-\xi}{2} - \frac{1}{4}$$

$$= \begin{cases} \dfrac{\xi - x}{2} - \dfrac{1}{4} & x < \xi, \\ \dfrac{x-\xi}{2} - \dfrac{1}{4} & x > \xi. \end{cases}$$

Note: The construction worked because the completely homogeneous problem has only the trivial solution, as can be easily checked.

# Solution Formula for the General Problem

Theorem. If the completely homogeneous problem $(L, B_1, B_2)$ has only the trivial solution, the problem with data $\{f; \gamma_1, \gamma_2\}$ has the unique solution

$$u(x) = \int_a^b g(x, \xi) f(\xi)\, d\xi + \frac{\gamma_2}{B_2 u_1} u_1(x) + \frac{\gamma_1}{B_1 u_2} u_2(x).$$

Proof. We have seen in the study of initial value problems that the integral satisfies the inhomogeneous differential equation while $u_1$ and $u_2$ solve the homogeneous equation. Thus, the sum solves $Lu = f$.

From

$$B_1 g = B_2 g = 0, \qquad B_1 u_1 = 0, \qquad B_2 u_2 = 0$$

we see that $u$ satisfies $B_u = \gamma_1$ and $B_2 u = \gamma_2$.