Second-Order Boundary Value Problems with Separated Boundary Conditions

Green's Function for Unmixed Boundary Conditions

We consider (L, B_1, B_2) with

$$B_1 u = \alpha_{11} u(a) + \alpha_{12} u'(a)$$
$$B_2 u = \beta_{21} u(b) + \beta_{22} u'(b)$$

Major Assumption: We suppose that the fully homogeneous problem has only the trivial solution.

Our goal is to find Green's function satisfying

$$\begin{aligned} Lg &= \delta(x - \xi), \qquad x, \xi \in (a, b), \\ B_1 g &= 0, \\ B_2 g &= 0. \end{aligned}$$

Two Basic Functions

Let u_1 satisfy the initial value problem

$$Lu_1 = 0, \qquad u_1(a) = \alpha_{12}, \qquad u_1'(a) = -\alpha_{11}.$$

Then u_1 satisfies

$$Lu_1 = 0, \qquad B_1 u_1 = 0.$$

Similarly, we can find u_2 such that

$$Lu_2 = 0, \qquad B_2 u_2 = 0.$$

From the Major Assumption, it follows that u_1 and u_2 must be independent.

Construction of Green's Function

Green's function has the form

$$g(x,\xi) = \begin{cases} c_1 \cdot u_1(x) & x < \xi, \\ c_2 \cdot u_2(x) & x > \xi \end{cases}$$

for some $c_1, c_2 \in \mathbb{R}$.

The continuity of g and the jump condition at $x = \xi$ give

$$\begin{aligned} c_1 \cdot u_1(\xi) - c_2 \cdot u_2(\xi) &= 0, \\ -c_1 \cdot u_1'(\xi) + c_2 \cdot u_2'(\xi) &= \frac{1}{a_2(\xi)}. \end{aligned}$$

Construction of Green's Function

Since u_1 and u_2 are independent, the Wronskian satisfies

$$W(u_1, u_2; \xi) \neq 0$$

Hence, by Cramer's rule,

$$c_1 = \frac{u_2(\xi)}{a_2(\xi)W(u_1, u_2; \xi)}, \qquad c_2 = \frac{u_1(\xi)}{a_2(\xi)W(u_1, u_2; \xi)}.$$

In summary, we see that

$$g(x, \xi) = \begin{cases} \dfrac{u_1(x)u_2(\xi)}{a_2(\xi)W(u_1, u_2; \xi)} & x < \xi, \\ \dfrac{u_1(\xi)u_2(x)}{a_2(\xi)W(u_1, u_2; \xi)} & x > \xi. \end{cases}$$

Construction of Green's Function

For short, we write

$$x_< := \min\{x, \xi\}, \qquad x_> := \max\{x, \xi\}$$

so

$$g(x, \xi) = \frac{u_1(x_<)u_2(x_>)}{a_2(\xi)W(u_1, u_2; \xi)}.$$

If $L = L^*$ there exists a constant $c \in \mathbb{C}$ such that

$$g(x, \xi) = c \cdot u_1(x_<)u_2(x_>)$$

Non-Homogeneous Boundary Conditions

Given that u_1 and u_2 satisfy

$$Lu_1 = 0, \qquad Lu_2 = 0, \qquad B_1 u_1 = 0, \qquad B_2 u_2 = 0,$$

we also see that

$$v(x) = \frac{\gamma_2}{B_2 u_1} u_1(x) + \frac{\gamma_1}{B_1 u_2} u_2(x)$$

satisfies

$$Lv = 0, \qquad B_1 v = \gamma_1, \qquad B_2 v = \gamma_2.$$