Second-Order Boundary Value Problems

The Second-Order Equation

We consider

$$Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u = f \qquad \text{on } (a, b) \subset \mathbb{R}$$

where

- f is piecewise continuous on $[a, b]$,
- $a_0, a_1, a_2 \in C([a, b])$,
- $a_p(x) \neq 0$ for all $x \in [a, b]$.

As usual, a classical solution

- is continuous on $[a, b]$,
- is continuously differentiable on (a, b),
- is twice differentiable and satisfies $Lu = f$ at all points in (a, b) where where f is continuous.

Boundary Conditions

We impose the boundary conditions

$$B_1 u := \alpha_{11} u(a) + \alpha_{12} u'(a) + \beta_{11} u(b) + \beta_{12} u'(b) = \gamma_1,$$
$$B_2 u := \alpha_{21} u(a) + \alpha_{22} u'(a) + \beta_{21} u(b) + \beta_{22} u'(b) = \gamma_2,$$

where $\alpha_{ij}, \beta_{ij}, \gamma_j \in \mathbb{R}$, $i, j = 1, 2$, and the row vectors

$$(\alpha_{11}, \alpha_{12}, \beta_{11}, \beta_{12}) \qquad \text{and} \qquad (\alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22})$$

are assumed to be independent.

B_1 and B_2 are called boundary functionals.

We say that $\{f; \gamma_1, \gamma_2\}$ is the data for the boundary value problem (L, B_1, B_2).

Types of Boundary Conditions

- Homogeneous boundary conditions: $\gamma_1 = \gamma_2 = 0$
- Fully homogeneous boundary value problem: data $\{0; 0, 0\}$
- Unmixed or separated boundary conditions:

$$B_1 u = \alpha_{11} u(a) + \alpha_{12} u'(a) = \gamma_1,$$
$$B_2 u = \beta_{21} u(b) + \beta_{22} u'(b) = \gamma_2,$$

- Initial conditions:

$$B_1 u = u(a) = \gamma_1,$$
$$B_2 u = u'(a) = \gamma_2.$$

Superposition principle

Suppose

- u solves (L, B_1, B_2) with data $\{f; \gamma_1, \gamma_2\}$
- $\widetilde{u}$ solves (L, B_1, B_2) with data $\{\widetilde{f}; \widetilde{\gamma}_1, \widetilde{\gamma}_2\}$

Then

- $c_1 u + c_2 \widetilde{u}$ solves (L, B_1, B_2) with data

$$\{c_1 f + c_2 \widetilde{f}; c_1 \gamma_1 + c_2 \widetilde{\gamma}_1, c_1 \gamma_1 + c_2 \widetilde{\gamma}_2\}$$

Existence and Uniqueness

- If the problem with data $\{0; 0, 0\}$ has only the trivial solution $u \equiv 0$, then the problem $\{f; \gamma_1, \gamma_2\}$ will have at most one classical solution.
- If the problem with data $\{0; 0, 0\}$ has a non-trivial solution, then the problem $\{f; \gamma_1, \gamma_2\}$ will have either no classical solution or an infinite number of classical solutions.

Major Assumption: Unless otherwise stated, we will always suppose that the problem with data $\{0; 0, 0\}$ has only the trivial solution.

Example of Non-Uniqueness / Non-Existence

$$-u'' = f(x), \quad 0 < x < 1, \qquad u'(0) = \gamma_1, \qquad u'(1) = \gamma_2.$$

The problem with data $\{0; 0, 0\}$ has the non-trivial solution $u(x) = 1$.

There can only be a solution of this problem if

$$\int_0^1 f(x)\, dx = \gamma_1 - \gamma_2.$$

- For data $\{1; 0, 0\}$ there are no classical solutions.
- For data $\{\sin(2\pi x); 0, 0\}$ there is an infinite number of solutions:

$$u(x) = C - \frac{x}{2\pi} + \frac{1}{4\pi^2}\sin(2\pi x), \qquad C \in \mathbb{R}.$$

Fundamental Solution

A fundamental solution $E(x,\xi)$ for L with pole at $\xi \in [a,b]$ satisfies

$$LE = \delta(x-\xi), \qquad\qquad x,\xi \in (a,b)$$

in the distributional sense.

We can construct a fundamental solution by imposing

- $LE = 0$ for $a < x < \xi$ and $\xi < x < b$
- E continuous on $[a,b]$, including at $x = \xi$
- Jump condition: $\lim_{\varepsilon \searrow 0} \left(\frac{dE}{dx}\Big|_{x=\xi+\varepsilon} - \frac{dE}{dx}\Big|_{x=\xi-\varepsilon} \right) = \frac{1}{a_2(\xi)}$

Green's Function

Green's function $g(x, \xi)$ for (L, B_1, B_2) is defined by the following properties

- $g(\cdot, \xi)$ is a fundamental solution with pole at $\xi \in (a, b)$
- $B_1 g = B_2 g = 0$

Since the difference of any two such functions has a continuous first derivative at $x = \xi$ and satisfies the problem with data $\{0; 0, 0\}$ (which has only the trivial solution), Green's function is uniquely defined, if it exists at all.

We write

$$Lg = \delta(x - \xi), \qquad x, \xi \in (a, b), \qquad B_1 g = 0, \qquad B_2 g = 0.$$