The Inhomogeneous Equation

Ordinary Differential Equations

We now discuss the inhomogeneous equation

$$Lu = f \qquad\qquad \text{on an open interval } I \subset \mathbb{R}$$

where

$$L = a_p(x)\frac{d^p}{dx^p} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)$$

and

- f is piecewise continuous on the closure $\overline{I}$ of I,
- $a_0, a_1, \ldots, a_p \in C(\overline{I})$,
- $a_p(x) \neq 0$ for all $x \in I$.

The Function u_ξ

Recall how to construct a causal fundamental solution for L:

Take $I = \mathbb{R}$ and fix $\xi \in \mathbb{R}$. Define u_ξ to satisfy

$$Lu_\xi = 0 \quad \text{on } \mathbb{R}$$

with initial values

$$u_\xi(\xi) = 0, \qquad \dots, \qquad u_\xi^{(p-2)}(\xi) = 0, \qquad u_\xi^{(p-1)}(\xi) = \frac{1}{a_p(\xi)}.$$

u_ξ is the solution of the IVP for L on $I = \mathbb{R}$ with data

$$\left\{0; 0, \dots, 0, \frac{1}{a_p(\xi)}\right\}_\xi .$$

Interpretation of u_ξ

We set

$$E(x,\xi) := H(x-\xi)u_\xi(x), \tag{I.1}$$

where H is the Heaviside function.

If the coefficient functions are smooth, $a_1 \dots, a_p \in C^\infty(\mathbb{R})$, we can interpret L as acting on the distribution E and find

$$LE = \delta(x-\xi).$$

Solution Formula for the Inhomogeneous Equation

We would therefore expect that the solution of

$$Lu = f \quad \text{on } \mathbb{R}, \qquad u(x_0) = 0, \qquad \dots, \qquad u^{(p-1)}(x_0) = 0,$$

is given by

$$u(x) = \int_{x_0}^{\infty} E(x,\xi) f(\xi)\, d\xi = \int_{x_0}^{x} u_\xi(x) f(\xi)\, d\xi.$$

Note: By the chain rule,

$$u'(x) = u_x(x) f(x) + \int_{x_0}^{x} u'_\xi(x) f(\xi)\, d\xi$$

where of course

$$u_x(x) = u_\xi(x)\big|_{\xi = x}.$$

Verification of the Solution Formula

Since $u_\xi(\xi) = 0$ for any $\xi \in \mathbb{R}$, we have

$$\begin{aligned} u(x) &= \int_{x_0}^{x} u_\xi(x) f(\xi)\, d\xi, \\ u'(x) &= \underbrace{u_x(x)}_{=0} f(x) + \int_{x_0}^{x} u'_\xi(x) f(\xi)\, d\xi \\ &= \int_{x_0}^{x} u'_\xi(x) f(\xi)\, d\xi \end{aligned}$$

and

$$u(x_0) = u'(x_0) = 0.$$

Verification of the Solution Formula

We continue to differentiate, yielding

$$u^{(p-1)}(x) = \underbrace{u_x^{(p-2)}(x)}_{=0} f(x) + \int_{x_0}^{x} u_\xi^{(p-1)}(x) f(\xi)\, d\xi,$$

so that u satisfies the initial conditions

$$\begin{aligned} u(x_0) &= 0, \\ u'(x_0) &= 0, \\ &\vdots \\ u^{(p-1)}(x_0) &= 0. \end{aligned}$$

Verification of the Solution Formula

Finally, at all points $x \in I$ where f is continuous,

$$
\begin{aligned}
u^{(p)}(x) &= \underbrace{u_x^{(p-1)}(x)}_{=1/a_p(x)} f(x) + \int_{x_0}^{x} u_\xi^{(p)}(x) f(\xi)\, d\xi \\
&= \frac{f(x)}{a_p(x)} + \int_{x_0}^{x} u_\xi^{(p)}(x) f(\xi)\, d\xi.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
Lu &= a_p(x) u^{(p)}(x) + \cdots + a_0(x) u(x) \\
&= f(x) + \int_{x_0}^{x} \underbrace{(Lu_\xi)(x)}_{=0} f(\xi)\, d\xi \\
&= f(x).
\end{aligned}
$$

The Solution Formula

Theorem. The unique classical solution of the initial value problem with data

$$\{f; 0, \ldots, 0\}_{x_0}$$

is given by

$$u(x) = \int_{x_0}^{x} u_\xi(x) f(\xi)\, d\xi$$

where u_ξ is the solution of the initial value problem with data

$$\{0; 0, \ldots, 0, 1/a_p(\xi)\}_\xi$$

If the coefficients $a_1, \ldots, a_p$ of L are constants, then

$$u_\xi(x) = u_0(x - \xi)$$

The Inhomogeneous Equation with General Data

The solution of the initial value problem with data

$$\{f; \gamma_1, \dots, \gamma_p\}_{x_0}$$

is given by

$$u(x) = \int_{x_0}^{x} u_\xi(x) f(\xi)\, d\xi + \gamma_1 u_1(x) + \cdots + \gamma_p u_p(x),$$

where u_k, $k = 1, \dots, p$, is a solution of the equation with data

$$\{0; 0, \dots, 0, \underset{\uparrow k}{1}, 0, \dots, 0\}_{x_0}$$