The Homogeneous Equation with Non-Vanishing Initial Conditions

Abel's Formula for the Wronskian

Suppose that $u_1, \dots, u_p$ are p solutions of

$$Lu = 0 \qquad \text{on } I \subset \mathbb{R}.$$

where L is given as in the previous section.

Then Abel's formula for the Wronskian is

$$W(u_1, \dots, u_p; x) = C \cdot e^{-m(x)} \qquad \text{for all } x \in I$$

where $C \in \mathbb{R}$ is some constant and m is a particular solution of

$$m'(x) = \frac{a_{p-1}(x)}{a_p(x)}.$$

Consequence of Abel's Formula

If $u_1, \dots, u_p$ are solutions of $Lu = 0$, then

$$W(u_1, \dots, u_p; x) = 0 \quad \text{for all } x \in I$$

if and only if

$$W(u_1, \dots, u_p; x_0) = 0 \quad \text{for a single } x_0 \in I.$$

Independence of Solutions

Theorem. Let $u_1, \dots, u_p$ be solutions of $Lu = 0$. Then

- $u_1, \dots, u_p$ are dependent

if and only if

- $W(u_1, \dots, u_p; x_0) = 0$ for some $x_0 \in I$.

Proof.
The Wronskian vanishes at a single point if and only if it vanishes everywhere on I.

If the solutions are dependent, then the Wronskian vanishes.

However, the converse is not obvious.

Independence of Solutions

Suppose that $W(u_1, \dots, u_p; x_0) = 0$.

Consider the system of equations

$$\begin{aligned}
u_1(x_0)y_1 \quad + \quad u_2(x_0)y_2 \quad + \quad \dots \quad + \quad u_p(x_0)y_p \quad &= \quad 0, \\
&\vdots \\
u_1^{(p-1)}(x_0)y_1 \quad + \quad u_2^{(p-1)}(x_0)y_2 \quad + \quad \dots \quad + \quad u_p^{(p-1)}(x_0)y_p \quad &= \quad 0,
\end{aligned}$$

for the p unknowns $y_1, \dots, y_p$.

Since $W(u_1, \dots, u_p; x_0) = 0$, this system has a non-trivial solution

$$(y_1, \dots, y_p) \in \mathbb{C}^p.$$

Independence of Solutions

Define

$$U(x) := y_1 u_1(x) + \cdots + y_p u_p(x).$$

Then $U(x)$ solves $Lu = 0$ with

$$\begin{aligned} U(x_0) &= 0 \\ U'(x_0) &= 0 \\ &\vdots \\ U^{(p-1)}(x_0) &= 0. \end{aligned}$$

Independence of Solutions

Since the solution of an initial value problem is unique,

$$U(x) = y_1 u_1(x) + \cdots + y_p u_p(x) = 0$$

for all $x \in I$, even though not all of the $y_k \in \mathbb{C}$ vanish.

Hence, the functions $(u_1, \ldots, u_p)$ are dependent.

Basis of Solutions

I.1. Theorem. Let $u_1, \dots, u_p$ be solutions to the initial value problem for L on I with data

- $\{0; 1, 0, \dots, 0\}_{x_0}$ in the case of u_1,
- $\{0; 0, 1, 0, \dots, 0\}_{x_0}$ in the case of u_2,

⋮

- $\{0; 0, \dots, 0, 1\}_{x_0}$ in the case of u_n.

Then $\{u_1, \dots, u_p\}$ is an independent set.

Any solution of $Lu = 0$ on I can be written in the form

$$u(x) = c_1 u_1(x) + \dots + c_p u_p(x)$$

for some $c_1, \dots, c_p \in \mathbb{C}$.

Basis of Solutions

The set is independent because $W(u_1, \dots, u_p; x_0) = 1 \neq 0$.

Any solution u_0 of $Lu = 0$ is completely determined by its initial values at some $x_0 \in \bar{I}$.

Since

$$u(x) := \underbrace{u_0(x_0)}_{=:c_1} u_1(x) + \underbrace{u_0'(x_0)}_{=:c_2} u_2(x) + \cdots + \underbrace{u_0^{(p-1)}(x_0)}_{=:c_p} u_p(x).$$

has just these initial values and solves $Lu = 0$,

$$u(x) = u_0(x).$$

This gives the desired representation.