Initial Value Problems, Independence and the Wronksian

Ordinary Differential Equations

Consider

$$Lu = f \qquad \text{on an open interval } I \subset \mathbb{R}$$

where

$$L = a_p(x)\frac{d^p}{dx^p} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)$$

and

- f is piecewise continuous on the closure $\bar{I}$ of I,
- $a_0, a_1, \ldots, a_p \in C(\bar{I})$,
- $a_p(x) \neq 0$ for all $x \in I$.

Initial Value Problems

Definition. An initial value problem (IVP) for L on I consists of the equation

$$Lu = f \qquad\qquad \text{on } I$$

and initial conditions at a point $x_0 \in \overline{I}$ given by

$$u(x_0) = \gamma_1, \qquad u'(x_0) = \gamma_2, \qquad \dots, \qquad u^{(p-1)}(x_0) = \gamma_p.$$

for some numbers $\gamma_1, \dots, \gamma_p \in \mathbb{R}$.

The data for the IVP is summarized by writing

$$\{f; \gamma_1, \gamma_2, \dots, \gamma_p\}_{x_0}.$$

Classical Solutions

Recall that a classical solution of the ODE

- is continuous on $\bar{I}$,
- is $p-1$ times continuously differentiable on I,
- is p times differentiable for all $x \in I$ where f is continuous,
- satisfies $Lu = f$ at all points in I where where f is continuous.

Theorem. The initial value problem

$$\begin{aligned} Lu &= f \quad \text{on } I, \\ u(x_0) &= \gamma_1, \\ &\vdots \\ u^{(p-1)}(x_0) &= \gamma_p, \end{aligned}$$

has a unique classical solution on $\bar{I}$.

Existence and Uniqueness of Solutions

The condition $a_p(x) \neq 0$ on $\bar{I}$ is essential.

Examples.

- The initial value problem

$$xu' - 2u = 0, \quad x \in \mathbb{R}, \qquad u(0) = 0$$

has more than one solution.

- The initial value problem

$$xu' + u = 0, \quad x \in \mathbb{R}, \qquad u(0) = 0$$

has no solution.

Linear Independence

Definition. A family $\{f_k\}_{k=1}^n$ of functions $f_1, \ldots, f_n \colon I \to \mathbb{C}$ is said to be (linearly) independent if

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \qquad \text{for all } x \in I$$

with $c_1, \ldots, c_n \in \mathbb{C}$ implies

$$c_1 = c_2 = \cdots = c_n = 0.$$

If $\{f_k\}_{k=1}^n$ is not independent, we say that the family is (linearly) dependent.

The Wronskian

Definition. For $f_1, \ldots, f_n \in C^{(p-1)}(I)$

$$W(f_1, \ldots, f_n; x) = \det \begin{pmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{pmatrix}$$

is called the Wronskian of $\{f_k\}_{k=1}^n$.

Note:

If $\{f_k\}_{k=1}^n$ is dependent, then $W(f_1, \ldots, f_n; x) = 0$.

The converse is in general false!

The Wronskian

Example. Suppose $f_1, f_2 \colon (-1, 1) \to \mathbb{R}$ are given by

$$f_1(x) = x^2, \qquad\qquad f_2(x) = |x| \cdot x.$$

Then f_1 and f_2 are independent, but

$$W(f_1, f_2; x) = 0 \qquad\qquad \text{for all } x \in (-1, 1).$$