Fundamental Solutions

Distributional Solutions

The most general problem for a differential operator L on a domain $\Omega \subset \mathbb{R}^n$ is

$$LT = S \qquad\qquad \text{on } \Omega$$

with given $S \in \mathcal{D}'(\mathbb{R}^n)$.

$T \in \mathcal{D}'(\mathbb{R}^n)$ is said to be a distributional solution if

$$(LT)(\varphi) = S\varphi$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \in \Omega$.

Note: If S is a regular distribution, then a regular distributional solution T is also a weak solution.

Fundamental Solutions

Definition. Let $\xi \in \mathbb{R}^n$ be fixed. A solution $E(\,\cdot\,;\xi) \in \mathcal{D}'(\mathbb{R}^n)$ of

$$LE(x;\xi) = \delta_\xi(x) = \delta(x-\xi)$$

is said to be a a fundamental solution for L with pole at ξ.

Note:

(i) E is a distributional solution of $LE = \delta(x-\xi)$. Often, but always, E is a locally integrable function..

(ii) Fundamental solutions are not unique; they may differ by addition of a solution of $Lu = 0$.

(iii) If the operator L has constant coefficients, then

$$E(x;\xi) = E(x-\xi;0).$$

Fundamental Solutions

Example. We have seen that

$$\Delta\left(\frac{1}{4\pi}\frac{1}{|x|}\right) = \delta(x) \qquad \text{in } \mathbb{R}^3.$$

Hence,

$$E(x;\xi) = \frac{1}{4\pi}\frac{1}{|x-\xi|}$$

is a fundamental solution with pole at ξ.

The same is true for

$$E(x;\xi) + u(x)$$

where u is any solution of $\Delta u = 0$, e.g.,

$$u(x_1, x_2, x_3) = x_1 x_2.$$

Causal Fundamental Solutions

Definition. Let

$$L = a_p(t)\frac{d^p}{dt^p} + \cdots + a_1(t)\frac{d}{dt} + a_0(t)$$

where $a_0, a_1, \dots, a_p$ are continuous functions defined on $\mathbb{R}$.

A fundamental solution $E(\,\cdot\,;\xi)\colon \mathbb{R} \to \mathbb{C}$ with pole at ξ is said to be *causal* if

$$E(x,\xi) = 0 \qquad\qquad \text{for } x < \xi.$$

Heuristic Construction

Suppose $E(t;\tau)$ is a causal fundamental solution with pole at τ, i.e.,

$$LE = a_p(t)\frac{d^pE}{dt^p} + \cdots + a_1(t)\frac{dE}{dt} + a_0(t)E = \delta(t-\tau)$$

and $E(t;\tau) = 0$ for $t < \tau$.

Assumption. $a_p(\tau) \neq 0$.

Here $E \in \mathcal{D}'(\mathbb{R})$. Let $E_{\text{Prim}} \in \mathcal{D}'(\mathbb{R})$ be a primitive of E, i.e., a distribution such that

$$E'_{\text{Prim}} = E.$$

(It can be shown that for any E such a distribution exists.)

Heuristic Construction

Suppose that E_{Prim} satisfies

$$a_p(t)\frac{d^p E_{\text{Prim}}}{dt^p} + \cdots + a_0(t)E_{\text{Prim}} = H(t-\tau).$$

Then $LE = \delta(t-\tau)$ and E is a fundamental solution.

The right-hand side is a locally integrable function which is discontinuous only at $t=\tau$.

We expect a classical solution E_{Prim}, i.e.,

$$E_{\text{Prim}} \in C^{(p-1)}(\mathbb{R}) \cap C^p(\mathbb{R}\setminus\{\tau\})$$

We also suppose

$$E_{\text{Prim}}(t,\tau) = 0 \qquad\qquad \text{for } t<\tau.$$

Heuristic Construction

Then for any $t < \tau$

$$E_{\text{Prim}}(t;\tau) = E'_{\text{Prim}}(t;\tau) = \cdots = E^{(p-1)}_{\text{Prim}}(t;\tau) = 0.$$

Since E_{Prim} is a classical solution,

$$E_{\text{Prim}}(\tau;\tau) = E'_{\text{Prim}}(\tau;\tau) = \cdots = E^{(p-1)}_{\text{Prim}}(\tau;\tau) = 0.$$

This implies

$$E(\tau;\tau) = E'(\tau;\tau) = \cdots = E^{(p-2)}(\tau;\tau) = 0.$$

We need one more initial condition.

Heuristic Construction

We divide

$$LE = a_p(t)\frac{d^p E}{dt^p} + \cdots + a_1(t)\frac{dE}{dt} + a_0(t)E = \delta(t-\tau)$$

by a_p, integrate and taking the limit:

On the right-hand side

$$\lim_{\varepsilon\to 0}\int_{\tau-\varepsilon}^{\tau+\varepsilon} \frac{1}{a_p(t)}\delta(t-\tau)\,dt = \frac{1}{a_p(\tau)}$$

On the left, by continuity,

$$\lim_{\varepsilon\to 0}\int_{\tau-\varepsilon}^{\tau+\varepsilon} \left(\frac{d^p E}{dt^p} + \cdots + \frac{a_1(t)}{a_p(t)}\frac{dE}{dt} + \frac{a_0(t)}{a_p(t)}E\right)\,dt = E^{(p-1)}(\tau;\tau)$$

Candidate for the Causal Fundamental Solution

We expect that, at least for $t > \tau$, $E(t;\tau)$ coincides with the solution $u_\tau(t)$ of

$$a_p(t)\frac{d^p u_\tau}{dt^p} + \cdots + a_1(t)\frac{du_\tau}{dt} + a_0(t)u_\tau = 0$$

with initial conditions

$$u_\tau(\tau) = u_\tau'(\tau) = \cdots = u_\tau^{(p-2)}(\tau) = 0, \qquad u_\tau^{(p-1)}(\tau) = \frac{1}{a_p(\tau)}.$$

We hence define the candidate

$$E(t;\tau) := H(t-\tau)u_\tau(t)$$

for the causal fundamental solution. We need to verify that

$$LE(t;\tau) = \delta(t-\tau)$$

Verification of the Causal Fundamental Solution

By Green's formula

$$\begin{aligned} LT_E\varphi = T_E(L^*\varphi) &= \int_{-\infty}^{\infty} H(t-\tau)u_\tau(t)L^*\varphi\,d\varphi \\ &= \int_{\tau}^{\infty} u_\tau(t)L^*\varphi\,d\varphi \\ &= \int_{\tau}^{\infty} \underbrace{(Lu_\tau)(t)}_{=0}\varphi(t)\,dt + J(u_\tau,\varphi)\Big|_{t=\tau}^{t=\infty}. \end{aligned}$$

Recall that

$$J(u_\tau,\varphi) = \sum_{k=1}^{p}\sum_{i+j=k-1}(-1)^i D^i(a_k\varphi)D^j u_\tau.$$

Since $\varphi \in C_0^\infty(\mathbb{R})$, J vanishes at infinity.

Verification of the Causal Fundamental Solution

All derivatives of u_τ of order less that $p-1$ vanish at $t=\tau$, so

$$J(u_\tau, \varphi)|_{t=\tau} = a_p(\tau)\varphi(\tau)\underbrace{D^{p-1}u_\tau(\tau)}_{=1/a_p(\tau)} = \varphi(\tau),$$

and hence

$$LT_E\varphi = \varphi(\tau),$$

as desired.

We have hereby established a method for finding causal fundamental solutions for ordinary differential operators.