# Classical and Weak Solutions

# Classical Solutions

Consider the differential equation

$$Lu = f \qquad \text{on } \Omega$$

where

- $L$ is an ordinary or partial differential operator
- $\Omega$ is a domain in $\mathbb{R}^n$
- $f$ is a continuous function on $\Omega$.

A classical solution is a function $u \in C^p(\Omega)$ such that

$$Lu = f \qquad \text{on } \Omega$$

in the usual sense.

# Weak Solutions

Now let

$$Lu = f \qquad \text{on } \Omega$$

where $f \in L^1_{\text{loc}}(\Omega)$, i.e.,

$$\int_B |f(x)|\,dx < \infty \qquad \text{for any bounded set } B \subset \Omega$$

A weak solution is a function $u \in L^1_{\text{loc}}(\Omega)$ such that

$$(LT_u)(\varphi) = T_f\varphi$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset \Omega$.

# Example: $xu'(x) = 0$ on $\mathbb{R}$

All classical solutions have the form

$$u(x) = c, \qquad\qquad c \in \mathbb{R}$$

We show that the Heaviside function

$$H(x) = \begin{cases} 1 & x \geq 0, \\ 0 & x < 0 \end{cases}$$

is a weak solution.

# Example: $xu'(x) = 0$ on $\mathbb{R}$

For any $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned}
xT'_H(\varphi) &= T'_H(x\varphi) \\
&= -T_H((x\varphi)') \\
&= -\int_{\mathbb{R}} H(x)\big(x\varphi(x)\big)'\,dx \\
&= -\int_0^\infty \big(\varphi(x) + x\varphi'(x)\big)\,dx \\
&= -\int_0^\infty \varphi(x)\,dx - x\varphi(x)\big|_0^\infty + \int_0^\infty \varphi(x)\,dx \\
&= 0
\end{aligned}$$

## Example: $\dfrac{\partial u}{\partial x_1}(x_1, x_2) = 0$ on $\mathbb{R}^2$

Any locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ that does not depend on $x_1$ is a weak solution, since

$$\begin{aligned}
\left(\frac{\partial}{\partial x_1} T_f\right)\varphi &= T_f\left(-\frac{\partial \varphi}{\partial x_1}\right) \\
&= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_2)\varphi_{x_1}(x_1, x_2)\, dx_1\, dx_2 \\
&= -\int_{-\infty}^{\infty} f(x_2) \underbrace{\int_{-\infty}^{\infty} \varphi_{x_1}(x_1, x_2)\, dx_1}_{=0}\, dx_2 \\
&= 0.
\end{aligned}$$

# Example: $u_{xx} - u_{tt} = 0$ on $\mathbb{R}^2$

d'Alembert's classical solution to the wave equation:

$$u(x,t) = f(x-t) + g(x+t)$$

for any $f, g \in C^2(\mathbb{R})$.

We show that

$$u(x,t) = H(x-t)$$

is a weak solution.

Note

$$L = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}, \qquad L^* = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} = L$$

# Example: $u_{xx} - u_{tt} = 0$ on $\mathbb{R}^2$

For any $\varphi \in \mathcal{D}(\mathbb{R}^2)$

$$(LT_{H(x-t)})(\varphi) = T_{H(x-t)}(L\varphi) = \int_{\mathbb{R}^2} H(x-t) L\varphi(x,t)\,dx\,dt = 0.$$

We perform a change of variables in the integral, setting

$$\begin{pmatrix} \xi \\ \tau \end{pmatrix} = \begin{pmatrix} x-t \\ x+t \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{=:A} \begin{pmatrix} x \\ t \end{pmatrix}.$$

We note that $\det A = 2$ and define $\widetilde{\varphi} \in \mathcal{D}(\mathbb{R}^2)$ by

$$\widetilde{\varphi}(\xi,\tau)\big|_{(\xi,\tau)=(x-t,x+t)} = \widetilde{\varphi}(x-t, x+t) := \varphi(x,t).$$

# Example: $u_{xx} - u_{tt} = 0$ on $\mathbb{R}^2$

Then

$$\varphi_{xx}(x,t) - \varphi_{tt}(x,t) = 4\widetilde{\varphi}_{\xi\tau}(\xi,\tau).$$

and

$$\begin{aligned}\int_{\mathbb{R}^2} H(x-t)L\varphi(x,t)\,dx\,dt &= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\xi)\widetilde{\varphi}_{\xi\tau}(\xi,\tau)\,d\xi\,d\tau \\ &= 2\int_{0}^{\infty}\int_{-\infty}^{\infty} \widetilde{\varphi}_{\xi\tau}(\xi,\tau)\,d\tau\,d\xi \\ &= 2\int_{0}^{\infty} \underbrace{\widetilde{\varphi}_{\xi}(\xi,\tau)\Big|_{-\infty}^{\infty}}_{=0}\,d\xi \\ &= 0\end{aligned}$$

which verifies the assertion.

# Classical and Weak Solutions

Lemma. Let $f \in C(\Omega)$. Then

(i) a classical solution of $Lu = f$ is also a weak solution.

(ii) a weak solution $u$ such that $u \in C^p(\Omega)$ is also a classical solution.

Proof.

(i) Let $u$ be a classical solution of $Lu = f$. Then for any $\varphi \in \mathcal{D}(\Omega)$,

$$(LT_u)(\varphi) = T_u(L^*\varphi) = \int_\Omega uL^*\varphi = \underbrace{-J(u,\varphi)|_{\partial\Omega}}_{\substack{=0 \text{ since} \\ \operatorname{supp}\varphi \subset \Omega}} + \int_\Omega \varphi Lu$$

$$= \int_\Omega f\varphi = T_f\varphi.$$

# Classical and Weak Solutions

(ii) Let $u \in C^p(\Omega)$ be a weak solution of $Lu = f$. Then for any $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega f\varphi = T_u(L^*\varphi) = \int_\Omega uL^*\varphi = \underbrace{-J(u,\varphi)|_{\partial\Omega}}_{=0} + \int_\Omega \varphi Lu,$$

so

$$\int_\Omega (Lu - f)\varphi = 0 \qquad\qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

We will show that this implies

$$Lu(x) = f(x) \qquad\qquad \text{for all } x \in \Omega$$

so $u$ is a classical solution.

# Classical and Weak Solutions

Suppose that

$$Lu(x_0) - f(x_0) > 0 \qquad \text{for some } x_0 \in \Omega$$

Since $Lu - f$ is continuous, there exists some neighborhood $B_\varepsilon(x_0)$ such that $Lu - f > 0$ on $B_\varepsilon(x_0)$.

We can find a cut-off function $\varphi \in C_0^\infty(B_\varepsilon(x_0))$ such that $\varphi \geq 0$ on $B_\varepsilon(x_0)$.

But then

$$\int_\Omega (Lu - f)\varphi > 0$$

which is a contradiction.

Thus, $Lu = f$ on $\Omega$. This completes the proof.