JOINT INSTITUTE 交大密西根学院

# The Equilibrium Heat Equation and Classical Solutions

# The Stationary Heat Equation

Heat equation: $\dfrac{\partial \theta}{\partial t} = \dfrac{\partial^2 \theta}{\partial x^2} + q(x,t), \qquad 0 < x < 1,\ t > 0.$

Suppose $q(x,t) = f(x)$ and search for an equilibrium solution

$$\theta(x,t) = u(x).$$

We obtain the Stationary Heat Equation

$$-\frac{d^2 u}{dx^2} = f(x), \qquad 0 < x < 1. \tag{I.1a}$$

We impose boundary conditions

$$u(0) = \alpha, \qquad u(1) = \beta, \qquad \alpha, \beta \in \mathbb{R}. \tag{I.1b}$$

# Data for the Equilibrium Heat Equation

The triple $(f, \alpha, \beta)$ is called the data for the equilibrium heat equation.

Superposition Principle: If

- $u_1$ satisfies (I.1) with data $(f_1, \alpha_1, \beta_1)$ and
- $u_2$ satisfies (I.1) with data $(f_2, \alpha_2, \beta_2)$

then

- $u_1 + u_2$ satisfies (I.1) with data $(f_1 + f_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)$.

Application:

Solution for $(f, 0, 0)$ + Solution for $(0, \alpha, \beta)$ = Solution for $(f, \alpha, \beta)$

# Classical Solutions

Consider the ODE

$$a_p(x)u^{(p)}(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x) = f(x)$$

on the interval $(a, b) \subset \mathbb{R}$ with

- $a_0, \ldots, a_p$ continuous on $[a, b]$
- $f$ piecewise continuous on $[a, b]$

A classical solution is a function $u$ such that

- $u$ is continuous on $[a, b]$
- $u$ is $p - 1$ times continuously differentiable on $(a, b)$
- $u$ is $p$ times differentiable at all points in $(a, b)$ where $f$ is continuous. At these points, $u$ solves the ODE.

## Classical Solutions

Thus, a classical solution to the boundary value problem

$$-\frac{d^2u}{dx^2} = 0, \qquad 0 < x < 1, \qquad u(0) = u(1) = 0$$

can not be something ridiculous like

$$u(x) = \begin{cases} 1 & 0 < x < 1. \\ 0 & x = 0 \text{ or } x = 1. \end{cases}$$

However: A classical solution does not have to be $p$-times differentiable at points where $f$ has jumps!

# Discontinuous Inhomogeneities

Example: Fix $0 < \xi < 1$ and consider the problem

$$-\frac{d^2u}{dx^2} = H_\xi(x), \qquad 0 < x < 1, \qquad u(0) = u(1) = 0,$$

with the Heaviside function

$$H_\xi(x) = \begin{cases} 0 & x < \xi \\ 1 & x > \xi \end{cases}$$

and we can define $H_\xi(\xi)$ any way we like.

Denote a solution of the boundary value problem by $u(x;\xi)$.

# Discontinuous Inhomogeneities

Solve separately in the intervals $(0, \xi)$ and $(\xi, 1)$:

$$u(x;\xi) = \begin{cases} Ax & 0 < x < \xi, \\ -(x-1)^2/2 + B(1-x) & \xi < x < 1, \end{cases}$$

with integration constants $A, B \in \mathbb{R}$.

Continuity of $u$ implies

$$\lim_{x \nearrow \xi} u(x;\xi) = \lim_{x \searrow \xi} u(x;\xi) \quad \Rightarrow \quad A\xi = -(\xi - 1)^2/2 + B(1-\xi).$$

Continuous differentiability implies

$$\lim_{x \nearrow \xi} u'(x;\xi) = \lim_{x \searrow \xi} u'(x;\xi) \qquad \Rightarrow \qquad A = -(\xi - 1) - B.$$

# Discontinuous Inhomogeneities

$$u(x;\xi) = \begin{cases} \frac{(\xi-1)^2}{2}x & 0 < x < \xi, \\ -(x-1)^2/2 + \frac{1-\xi^2}{2}(1-x) & \xi < x < 1. \end{cases}$$