Application of the Fourier Transform to Partial Differential Equations

The Convolution for Tempered Distributions

For $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ the convolution is defined by

$$(\varphi * \psi)(y) := \int_{\mathbb{R}^n} \varphi(y-x)\psi(x)\, dx.$$

Does not work for two distributions!

Definition. Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$. Then $T * \psi \in \mathcal{S}'(\mathbb{R}^n)$ is defined by

$$(T * \psi)(\varphi) := T(\widetilde{\psi} * \varphi),$$

where $\widetilde{\psi}(x) = \psi(-x)$.

If $T = T_g$ for some $g \in \mathcal{S}(\mathbb{R}^n)$,

$$T_g * \psi = T_{g*\psi}.$$

The Convolution for Tempered Distributions

Example. The convolution of the Dirac distribution with a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$\begin{aligned}(T_\delta * \psi)(\varphi) &= T_\delta\left(\int_{\mathbb{R}} \psi(-x)\varphi((\,\cdot\,) - x)\,dx\right)\\ &= \int_{\mathbb{R}} \psi(-x)\varphi(0 - x)\,dx\\ &= \int_{\mathbb{R}} \psi(x)\varphi(x)\,dx = T_\psi\varphi,\end{aligned}$$

so

$$\delta * \psi = \psi$$

Properties of the Convolution

For $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi, \chi \in \mathcal{S}(\mathbb{R}^n)$

(i) $D^\beta(T * \psi) = (D^\beta T) * \psi = T * D^\beta \psi$,

(ii) $(T * \psi) * \chi = T * (\psi * \chi)$

(iii) $\widehat{T * \psi} = (2\pi)^{n/2} \hat{\psi} \hat{T}$ where

$$\hat{\psi} \hat{T}(\varphi) = \hat{T}(\hat{\psi} \varphi).$$

Very useful for solving partial differential equations!

Example: The Heat Equation

Heat equation on $\mathbb{R}^n$:

$$\frac{\partial u}{\partial t} - \Delta u = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+.$$

Initial condition:

$$u(x,0) = f(x), \qquad f \in \mathcal{S}'(\mathbb{R}^n).$$

Assumption:

$$u(\,\cdot\,,t) \in \mathcal{S}'(\mathbb{R}^n) \qquad \text{for all } t \geq 0$$

Example: The Heat Equation

Treat $t \geq 0$ as a parameter and apply Fourier transform "with respect to the x-variable". Then

$$\frac{\partial \hat{u}}{\partial t} + |\xi|^2 \hat{u} = 0, \qquad (\xi, t) \in \mathbb{R}^n \times \mathbb{R}_+,$$

with initial condition

$$\hat{u}(\xi, 0) = \hat{f}(\xi)$$

Unique solution:

$$\hat{u}(\xi, t) = e^{-t|\xi|^2} \hat{f}(\xi)$$

Set

$$\hat{\psi}(\xi, t) := e^{-t|\xi|^2}$$

Example: The Heat Equation

Then

$$\hat{u}(\xi, t) = \hat{\psi}(\xi, t)\hat{f}(\xi)$$

By convolution properties, for $t > 0$,

$$u(x, t) = (2\pi)^{-n/2} f * \psi(\,\cdot\,, t)$$

From $\hat{\psi}(\xi, t) := e^{-t|\xi|^2}$,

$$\psi(x, t) = (2t)^{-n/2} e^{-|x|^2/(4t)}$$

Then

$$u(x, t) = f * p(\,\cdot\,, t)$$

where

$$p(x, t) := (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$$

(Heat kernel)

Example: The Heat Equation

Theorem. The heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \tag{I.1}$$

with initial condition

$$u(x,0) = f(x), \qquad f \in \mathcal{S}'(\mathbb{R}^n).$$

has the unique solution $u(\,\cdot\,,t) \in \mathcal{S}'(\mathbb{R}^n)$ given by

$$u(x,t) = f * p(x,t), \qquad t > 0,$$

where

$$p(x,t) := (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$$

Example: The Heat Equation

If $f \in \mathcal{S}(\mathbb{R}^n)$, then $u(\,\cdot\,, t) \in \mathcal{S}(\mathbb{R}^n)$ for all $t > 0$.

Furthermore,

$$u(x,t) = f * p(x,t) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} f(y) e^{-|x-y|^2/(4t)}\, dy$$

Since $p(\,\cdot\,, t)$ is a delta family as $t \searrow 0$, we see that

$$\lim_{t \searrow 0} u(x,t) = f(x),$$

as expected.

These formulas hold also if f is only continuous and bounded.

The uniqueness of the solution requires $u(\,\cdot\,, t) \in \mathcal{S}'(\mathbb{R}^n)$. There exist other solutions of the heat equation with initial condition that "blow up" at infinity.