Tempered Distributions

Tempered Distributions

A linear functional on $\mathcal{S}(\mathbb{R}^n)$ is a map $T\colon \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ such that

$$T(\lambda\varphi_1 + \mu\varphi_2) = \lambda T\varphi_1 + \mu T\varphi_2$$

for $\varphi_1, \varphi_2 \in \mathcal{D}$, $\lambda, \mu \in \mathbb{C}$.

T is said to be continuous if

$$\varphi_m \to 0 \qquad \Rightarrow \qquad T\varphi_m \to 0$$

A continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$ is called a tempered distribution on $\mathbb{R}^n$.

The set of all tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

$\mathcal{S}'(\mathbb{R}^n)$ is a vector space.

Tempered Distributions

Since

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$$

it is easy to see that

$$\mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$$

so every tempered distribution is also a distribution.

If $T_g \in \mathcal{S}'(\mathbb{R}^n)$ is given by

$$T_g\varphi := \int_{\mathbb{R}^n} g(x)\varphi(x)\,dx \qquad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n)$$

for some function g we simply write $g \in \mathcal{S}'(\mathbb{R}^n)$.

Examples of Tempered Distributions

- $T_\delta \colon \varphi \mapsto \varphi(0)$ is a tempered distribution on $\mathbb{R}^n$
- $g(x) = x^2$ is a tempered distribution on $\mathbb{R}$.
- $g(x) = e^{x^2}$ is not a tempered distribution on $\mathbb{R}$, since

$$\int_{-\infty}^{\infty} e^{x^2} \varphi(x)\, dx$$

  does not exist for all Schwartz functions φ, e.g., not for $\varphi(x) = e^{-x^2}$.

The term "tempered" refers to the growth of g at infinity, which can not be too rapid.

The Fourier Transform for Tempered Distributions

Definition. The Fourier transform of $T \in \mathcal{S}'(\mathbb{R}^n)$ is defined by

$$\hat{T}\varphi := T\hat{\varphi}.$$

where $\hat{\varphi}$ is the Fourier transform of the Schwartz function φ.

We also write

$$\mathcal{F}T \quad \text{for} \quad \hat{T}.$$

Remarks.

- Since $\hat{\varphi} \in \mathcal{S}$ if $\varphi \in \mathcal{S}$, the right-hand side is well-defined.
- Since the Fourier transform and T are continuous and linear, $\hat{T}$ will be continuous and linear. Therefore, $\hat{T} \in \mathcal{S}'(\mathbb{R}^n)$.

The Fourier Transform for Tempered Distributions

Since

$$\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$$

we check that our definition is compatible with the previous one for Schwartz functions: If $g \in \mathcal{S}(\mathbb{R}^n)$, then $T_g \in \mathcal{S}'(\mathbb{R}^n)$ and

$$\begin{aligned}
(\hat{T}_g)\varphi = T_g(\hat{\varphi}) &= \int_{\mathbb{R}^n} g(\xi)\hat{\varphi}(\xi)\,d\xi \\
&= \int_{\mathbb{R}^n} g(\xi)\cdot(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\varphi(x)\,dx\,d\xi \\
&= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(\xi)e^{-i\langle x,\xi\rangle}\,d\xi\,\varphi(x)\,dx \\
&= \int_{\mathbb{R}^n} \hat{g}(x)\varphi(x)\,dx \\
&= T_{\hat{g}}\varphi.
\end{aligned}$$

The Fourier Transform for Tempered Distributions

We have extended the Fourier transform to a continuous, bijective map

$$\mathcal{F}\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

The inverse is given by

$$(\mathcal{F}^{-1}T)(\varphi) = T(\mathcal{F}^{-1}\varphi).$$

Example.

$$\begin{aligned}\hat{T}_\delta\varphi &= T_\delta\hat{\varphi} = \hat{\varphi}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{e^{-ix\cdot 0}}_{=1}\varphi(x)\,dx \\ &= T_{1/\sqrt{2\pi}}\varphi\end{aligned}$$

so

$$\hat{\delta} = 1/\sqrt{2\pi}.$$

Example: $g \in \mathcal{S}'(\mathbb{R})$, $g(x) = 1$

Since $T_g = \sqrt{2\pi}\,\hat{T}_\delta$,

$$\hat{T}_g \varphi = T_g \hat{\varphi} = \sqrt{2\pi}\,\hat{T}_\delta \hat{\varphi} = \sqrt{2\pi}\,T_\delta \hat{\hat{\varphi}}$$

Since $\hat{\hat{\varphi}}(x) = \varphi(-x)$, we have

$$\hat{T}_g \varphi = \sqrt{2\pi}\varphi(-0) = \sqrt{2\pi}\,T_\delta \varphi$$

so

$$\hat{1} = \sqrt{2\pi}\delta$$

Formally,

$$\hat{1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, d\xi = \sqrt{2\pi}\delta(x)$$

which coincides with the Dirichlet kernel as a delta family.

Example: $g \in \mathcal{S}'(\mathbb{R})$, $g(x) = x$

$$
\begin{aligned}
\hat{T}_g \varphi = T_g \hat{\varphi} &= \int_{-\infty}^{\infty} \xi \cdot \hat{\varphi}(\xi)\, d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi e^{-ix\xi} \varphi(x)\, dx\, d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} i \frac{d}{dx} \big(e^{-ix\xi}\big) \varphi(x)\, dx\, d\xi \\
&= \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ix\xi} \varphi'(x)\, dx\, d\xi \\
&= \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} e^{-ix\xi}\, d\xi}_{=2\pi\delta(x)} \varphi'(x)\, dx \\
&= -i\sqrt{2\pi} \varphi'(0).
\end{aligned}
$$

Example: The Heaviside function $H \in \mathcal{S}'(\mathbb{R})$

$$
\begin{aligned}
\hat{T}_H\varphi = T_H\hat{\varphi} = \int_0^\infty \hat{\varphi}(\xi)\,d\xi &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \int_{-\infty}^\infty e^{-ix\xi}\varphi(x)\,dx\,d\xi \\
&= \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \varphi(x)\int_0^R e^{-ix\xi}\,d\xi\,dx \\
&= \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \varphi(x)\frac{e^{-iRx}-1}{-ix}\,dx \\
&= \lim_{R\to\infty} \frac{i}{\sqrt{2\pi}}\int_{-\infty}^\infty \varphi(x)\frac{\cos(Rx)-1}{x}\,dx \\
&\quad + \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \varphi(x)\frac{\sin(Rx)}{x}\,dx
\end{aligned}
$$

Example: The Heaviside function $H \in \mathcal{S}'(\mathbb{R})$

It can be shown that

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \varphi(x) \frac{\mathbf{1 - \cos(\mathit{Rx})}}{x}\, dx = \mathcal{P}\left(\frac{1}{x}\right)\varphi.$$

Furthermore,

$$\frac{\sin(Rx)}{\pi x}$$

is a delta family as $R \to \infty$ (the Dirichlet kernel again) so

$$\hat{H}(\xi) = \frac{-i}{\sqrt{2\pi}} \mathcal{P}\left(\frac{1}{\xi}\right) + \sqrt{\frac{\pi}{2}}\delta(\xi).$$