The Fourier Inversion Formula and Properties of the Fourier Transform

The Fourier Inversion Formula

Theorem. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\varphi(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{\varphi}(\xi) e^{i\langle x,\xi\rangle} \, d\xi$$

Proof for $n = 1$.

Suppose first that $\varphi \in \mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \hat{\varphi}(\xi) e^{ix\xi} \, d\xi &= \frac{1}{2\pi} \int_{-R}^{R} e^{ix\xi} \int_{-\infty}^{\infty} \varphi(\omega) e^{-i\omega\xi} \, d\omega \, d\xi \\
&= \int_{-\infty}^{\infty} \varphi(\omega) \frac{1}{2\pi} \int_{-R}^{R} e^{i(x-\omega)\xi} \, d\xi \, d\omega \\
&= \int_{-\infty}^{\infty} \varphi(x-y) \frac{1}{2\pi} \int_{-R}^{R} e^{iy\xi} \, d\xi \, dy
\end{aligned}$$

The Fourier Inversion Formula

Recall that the Dirichlet kernel

$$\frac{1}{2\pi}\int_{-R}^{R} e^{iy\xi}\,d\xi$$

is a delta family that converges to $\delta(y)$ as $R \to \infty$.

Therefore,

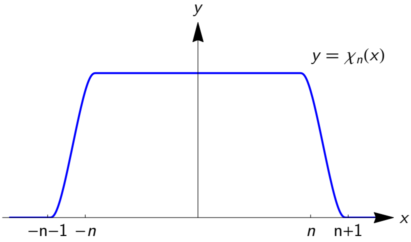
$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\varphi}(\xi)e^{ix\xi}\,d\xi &= \lim_{R\to\infty}\int_{-\infty}^{\infty}\varphi(x-y)\frac{1}{2\pi}\int_{-R}^{R} e^{iy\xi}\,d\xi\,dy \\ &= \varphi(x-0) \\ &= \varphi(x)\end{aligned}$$

This proves the statement for $\varphi \in \mathcal{D}(\mathbb{R})$.

The Fourier Inversion Formula

Let $\chi_n \in \mathcal{D}(\mathbb{R})$, $n \in \mathbb{N}$, be cut-off functions with

$$\chi_n(x) = \begin{cases} 1 & |x| < n \\ 0 & |x| > n+1 \end{cases}$$



Now suppose $\varphi \in \mathcal{S}(\mathbb{R})$. Then $\chi_n\varphi \in \mathcal{D}(\mathbb{R})$ and

$$\chi_n\varphi \xrightarrow{n\to\infty} \varphi$$

in $\mathcal{S}(\mathbb{R})$.

The Fourier Inversion Formula

The Fourier inversion formula states simply that

$$\hat{\hat{\varphi}}(-x) = \varphi(x) \qquad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}).$$

We have proven the inversion formula for all test functions, so

$$\widehat{\widehat{\chi_n\varphi}}(-x) = \chi_n\varphi(x) \qquad \text{for all } n \in \mathbb{N}.$$

Since the Fourier transform and the reflection $\varphi(x) \mapsto \varphi(-x)$ are continuous, we can let $n \to \infty$ on both sides, yielding

$$\hat{\hat{\varphi}}(-x) = \varphi(x).$$

This completes the proof.

Properties of the Fourier Transform

Suppose $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$.

(i) (Dilation) For $\alpha \in \mathbb{R}_+$ we define $D_\alpha \varphi(x) = \alpha^{n/2}\varphi(\alpha x)$. Then

$$\mathcal{F}(D_\alpha \varphi) = D_{1/\alpha}\mathcal{F}\varphi.$$

(ii) (Translation) For $y \in \mathbb{R}^n$ we define $\tau_y \varphi(x) = \varphi(x - y)$. Then

$$(\mathcal{F}\tau_y\varphi)(\xi) = e^{-i\langle y,\xi\rangle}\mathcal{F}\varphi(\xi).$$

(iii) (Unitarity) Let $\langle \varphi, \psi \rangle_{L^2} := \int_{\mathbb{R}^n} \overline{\varphi(x)}\psi(x)\,dx$. Then

$$\langle \hat{\varphi}, \hat{\psi} \rangle_{L^2} = \langle \varphi, \psi \rangle_{L^2}.$$

The Convolution

Definition. The convolution of $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$(\varphi * \psi)(y) := \int_{\mathbb{R}^n} \varphi(y - x)\psi(x)\, dx.$$

Properties. For $\varphi, \psi, \chi \in \mathcal{S}(\mathbb{R}^n)$,

i) $\varphi * \psi \in \mathcal{S}(\mathbb{R}^n)$

ii) $\varphi * \psi = \psi * \varphi$

iii) $\varphi * (\psi * \chi) = (\varphi * \psi) * \chi$

iv) $(2\pi)^{n/2}\widehat{\varphi \cdot \psi} = \hat{\varphi} * \hat{\psi}$

v) $\widehat{\varphi * \psi} = (2\pi)^{n/2}\hat{\varphi} \cdot \hat{\psi}$