Continuity of the Fourier Transform

Convergence and Continuity in $\mathcal{S}(\mathbb{R}^n)$

Definition. Let (φ_m) be a sequence with $\varphi_m \in \mathcal{S}(\mathbb{R}^n)$, $m \in \mathbb{N}$.

(i) (φ_m) is a null sequence in $\mathcal{S}(\mathbb{R}^n)$ if for all $\alpha, \beta \in \mathbb{N}^n$

$$\sup_{x \in \mathbb{R}} |x^\alpha D^\beta \varphi_m(x)| \xrightarrow{m\to\infty} 0.$$

(ii) A linear map $L\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is said to be continuous if

$$L\varphi_m \xrightarrow{m\to\infty} 0$$

for all null sequences (φ_m) in $\mathcal{S}(\mathbb{R}^n)$.

$\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is Linear and Continuous

Theorem. The Fourier transform is a continuous, linear map

$$\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n).$$

Proof for $n = 1$.

1) $\varphi \in \mathcal{S}(\mathbb{R}) \;\Rightarrow\; \hat{\varphi} \in \mathcal{S}(\mathbb{R})$

We need to show that $\hat{\varphi} \in C^\infty(\mathbb{R})$ and

$$\sup_{\xi \in \mathbb{R}} \left| \xi^j \frac{d^k \hat{\varphi}(\xi)}{d\xi^k} \right| < \infty$$

for all $j, k \in \mathbb{N}$.

$\mathcal{F}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is Linear and Continuous

$\hat{\varphi} \in C^\infty(\mathbb{R})$:

$$\frac{d^k}{d\xi^k}\hat{\varphi}(\xi) = [\widehat{(-ix)^k\varphi}](\xi)$$

The right-hand side exists since $(-ix)^k\varphi \in \mathcal{S}(\mathbb{R})$ for any $k \in \mathbb{N}$.

$\sup_{\xi\in\mathbb{R}} \left|\xi^j \frac{d^k\hat{\varphi}(\xi)}{d\xi^k}\right| < \infty$:

$$\begin{aligned}\left|(-i\xi)^j \frac{d^k\hat{\varphi}(\xi)}{d\xi^k}\right| &\le \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left|\frac{d^j}{dx^j}\big((ix)^k\varphi(x)\big)\right| \cdot |e^{-ix\xi}|\,dx \\ &\le \frac{1}{\sqrt{2\pi}}\sup_{x\in\mathbb{R}}\bigg|(1+x^2)\underbrace{\frac{d^j}{dx^j}\big((ix)^k\varphi(x)\big)}_{\in\mathcal{S}(\mathbb{R})}\bigg|\underbrace{\int_{\mathbb{R}}\frac{1}{1+x^2}\,dx}_{<\infty} \\ &< \infty\end{aligned}$$

$\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is Linear and Continuous

2) $\mathcal{F}$ is linear

For $\lambda, \mu \in \mathbb{C}$, $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and any $\xi \in \mathbb{R}$,

$$\begin{aligned}
\mathcal{F}[\lambda\varphi + \mu\psi](\xi) &= (2\pi)^{-n/2} \int_{\mathbb{R}} [\lambda\varphi(x) + \mu\psi(x)] e^{-ix\xi}\, dx \\
&= \lambda \cdot (2\pi)^{-n/2} \int_{\mathbb{R}} \varphi(x) e^{-ix\xi}\, dx \\
&\quad + \mu \cdot (2\pi)^{-n/2} \int_{\mathbb{R}} \psi(x) e^{-ix\xi}\, dx \\
&= \lambda(\mathcal{F}\varphi)(\xi) + \mu(\mathcal{F}\psi)(\xi)
\end{aligned}$$

$\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is Linear and Continuous

3) $\mathcal{F}$ is continuous

   We have seen that for some $C > 0$

$$\sup_{\xi\in\mathbb{R}} \left| \xi^j \frac{d^k \hat{\varphi}(\xi)}{d\xi^k} \right| \le C \cdot \sup_{x\in\mathbb{R}} \left| (1+x^2) \frac{d^j}{dx^j} \big((ix)^k \varphi(x) \big) \right|$$

   If (φ_m) is a null sequence, the right-hand side converges to zero. Therefore, the left-hand side converges to zero and $(\widehat{\varphi_m})$ is also a null sequence.

This completes the proof.