Delta Families

Delta Families and Delta Sequences

Definition. Suppose

- $I \subset \mathbb{R}$ (index set),
- $f_\alpha \in L^1_{\text{loc}}(\mathbb{R}^n)$ for all $\alpha \in I$

Then $\{f_\alpha\}_{\alpha \in I}$ is a delta family (as $\alpha \to \alpha_0$) if

$$\lim_{\alpha \to \alpha_0} f_\alpha = \delta.$$

If $I = \mathbb{N}$ and $\alpha_0 = \infty$ then $\{f_\alpha\}_{\alpha \in I}$ is a delta sequence.

Constructing Delta Families

Theorem. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

- $f(x) \geq 0$ for all $x \in \mathbb{R}^n$,
- $\int_{\mathbb{R}^n} f(x)\,dx = 1$

Then

$$f_\alpha(x) = \frac{1}{\alpha^n} f\left(\frac{x}{\alpha}\right) \qquad \text{for } \alpha > 0$$

defines a delta family $\{f_\alpha\}_{\alpha>0}$ as $\alpha \to 0$. In particular,

$$\lim_{\alpha \searrow 0} \int_{\mathbb{R}^n} f_\alpha(x)\varphi(x)\,dx = \varphi(0)$$

for any φ that is bounded and continuous at $x = 0$.

Constructing Delta Families

Proof.

Suppose $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is bounded and continuous at $x = 0$.

$$T_{f_\alpha}\varphi = \int_{\mathbb{R}^n} f_\alpha(x)\varphi(x)\,dx = \varphi(0) + \int_{\mathbb{R}^n} f_\alpha(x)\underbrace{(\varphi(x) - \varphi(0))}_{=:\psi(x)}\,dx$$

Prove:

$$\lim_{\alpha\to 0}\int_{\mathbb{R}^n} f_\alpha(x)\psi(x)\,dx = 0$$

Show that for every $\varepsilon > 0$ there exists a $\gamma > 0$ such that

$$\alpha < \gamma \quad \Rightarrow \quad \left|\int_{\mathbb{R}^n} f_\alpha(x)\psi(x)\,dx\right| < \varepsilon.$$

Constructing Delta Families

(i) For any $\alpha > 0$,

$$\int_{\mathbb{R}^n} f_\alpha(x)\,dx = 1$$

(ii) For any $R > 0$,

$$\lim_{\alpha \to 0} \int_{|x|>R} f_\alpha(x)\,dx = 0$$

(iii) For any $R > 0$,

$$\lim_{\alpha \to 0} \int_{|x|<R} f_\alpha(x)\,dx = 1$$

Constructing Delta Families

Since $f \geq 0$, for any $R > 0$,

$$\left|\int_{|x|<R} f_\alpha(x)\psi(x)\,dx\right| \leq \underbrace{\max_{|x|\leq R}|\psi(x)|}_{=:c(R)} \cdot \underbrace{\int_{|x|<R} f_\alpha(x)\,dx}_{\leq 1}.$$

Furthermore,

$$\left|\int_{|x|>R} f_\alpha(x)\psi(x)\,dx\right| \leq \underbrace{\sup_{x\in\mathbb{R}^n}|\psi(x)|}_{=:M} \cdot \int_{|x|>R} f_\alpha(x)\,dx.$$

Then

$$\left|\int_{\mathbb{R}^n} f_\alpha(x)\psi(x)\,dx\right| \leq c(R) + M\int_{|x|>R} f_\alpha(x)\,dx$$

Constructing Delta Families

Fix $\varepsilon > 0$.

- Choose $R > 0$ small enough so that $c(R) < \varepsilon/2$.
- Choose $\gamma > 0$ small enough so that

$$\left| \int_{|x|>R} f_\alpha(x)\, dx \right| < \frac{\varepsilon}{M} \qquad \text{for } \alpha < \gamma$$

Then

$$\left| \int_{\mathbb{R}^n} f_\alpha(x)\psi(x)\, dx \right| < \varepsilon$$

for all $\alpha < \gamma$.

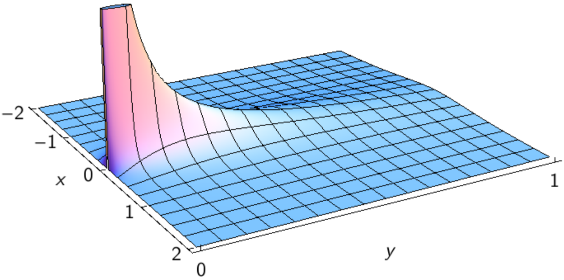
This completes the proof.

Example: $f(x) = \frac{1}{\pi(x^2+1)}$

$$f_y(x) = \frac{1}{y} f\left(\frac{x}{y}\right) = \frac{y}{\pi(x^2 + y^2)}, \qquad y > 0,$$

with

$$f_y(x) \to \delta(x) \qquad \text{as } y \searrow 0$$

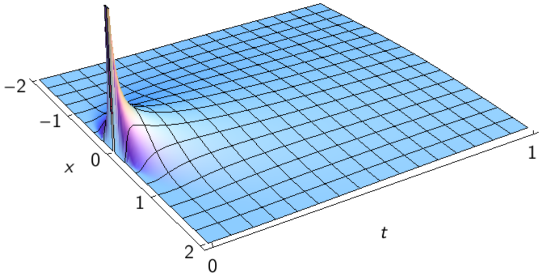


Example: $f(x) = \frac{1}{\sqrt{4\pi}} e^{-x^2/4}$

$$f_t(x) = \frac{1}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}, \qquad t > 0,$$

with

$$f_t(x) \to \delta(x) \qquad \text{as } t \searrow 0$$

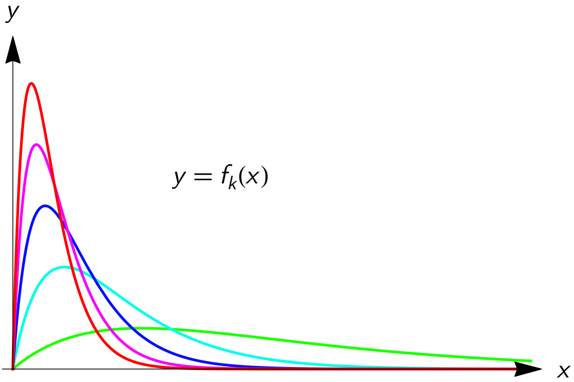


Example: $f(x) = H(x)xe^{-x}$

$$f_k(x) = k^2 H(x) x e^{-kx}, \qquad k \in \mathbb{N},$$

with

$$f_k(x) \to \delta(x) \qquad \text{as } k \to \infty$$

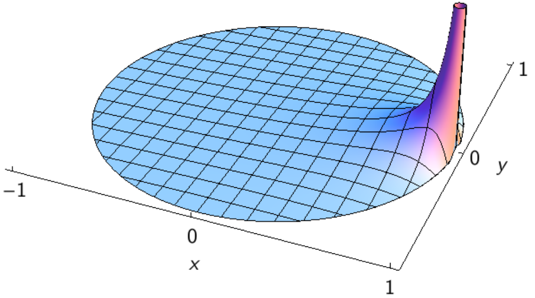


The Poisson Kernel

$$f_r(\theta) = \begin{cases} \frac{1}{2\pi} \cdot \frac{1-r^2}{1+r^2-2r\cos\theta} & |\theta| \le \pi, \\ 0 & |\theta| > \pi, \end{cases} \qquad 0 \le r < 1,$$

where

$$f_r(\theta) \to \delta(\theta) \qquad \text{as } r \nearrow 1$$



The Dirichlet Kernel

$$f_R(x) = \frac{1}{2\pi} \int_{-R}^{R} e^{i\omega x}\, d\omega = \frac{\sin(Rx)}{\pi x}, \qquad R > 0, \qquad \text{(I.1)}$$

where

$$f_R(x) \to \delta(x) \qquad \text{as } R \to \infty$$