Families of Distributions

Families of Functions

Example. The functions $f_n\colon \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \begin{cases} n & |x| < 1/(2n), \\ 0 & \text{otherwise}, \end{cases} \qquad n \in \mathbb{N},$$

define a sequence $(f_n)_{n\in\mathbb{N}\setminus\{0\}}$.

More generally, for any $\varepsilon > 0$ the functions $f_\varepsilon\colon \mathbb{R} \to \mathbb{R}$

$$f_\varepsilon(x) = \begin{cases} 1/\varepsilon & |x| < \varepsilon/2, \\ 0 & \text{otherwise}, \end{cases}$$

define a family of functions $(f_\varepsilon)_{\varepsilon>0}$.

Weak Convergence of Distributions

Definition. Suppose

- $I \subset \mathbb{R}$ index set
- $\{T_\alpha\}_{\alpha \in I}$ family of distributions, $T_\alpha \in \mathcal{D}'(\mathbb{R}^n)$
- $\alpha_0 \in \bar{I}$ in the index set or boundary point of index set.
- $T \in \mathcal{D}'(\mathbb{R}^n)$ given.

Then

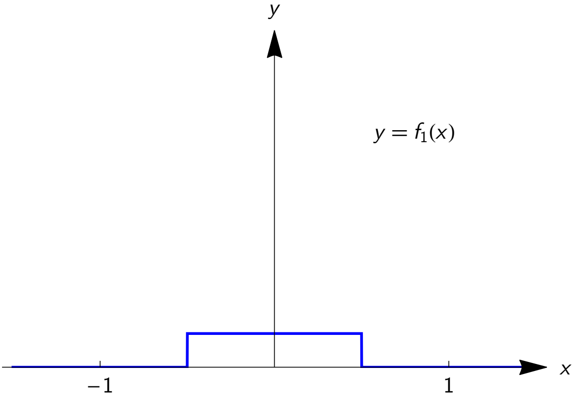
$$\lim_{\alpha \to \alpha_0} T_\alpha = T \qquad :\Leftrightarrow \qquad \lim_{\alpha \to \alpha_0} T_\alpha \varphi = T\varphi$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

(Weak Convergence or Distributional Convergence)

Example: Convergence to a Point Source

$$f_n\colon \mathbb{R} \to \mathbb{R} \qquad f_n(x) = \begin{cases} n & |x| < 1/(2n), \\ 0 & \text{otherwise.} \end{cases}$$



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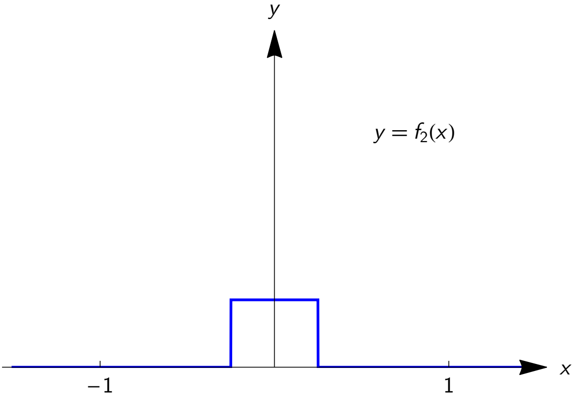

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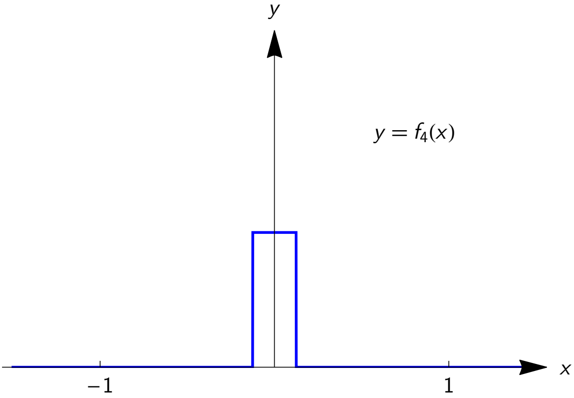

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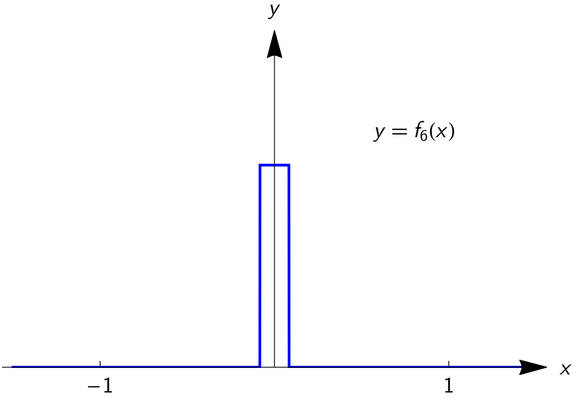

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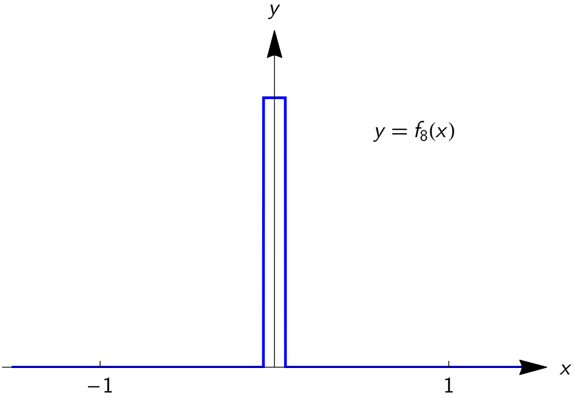

Weak Convergence of (f_n)

For any $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{R}} f_n(x)\varphi(x)\,dx &= n\int_{-1/(2n)}^{1/(2n)} \varphi(x)\,dx \\ &= \varphi(0) + n\int_{-1/(2n)}^{1/(2n)} (\varphi(x)-\varphi(0))\,dx\end{aligned}$$

Hence,

$$\begin{aligned}|T_{f_n}\varphi - \varphi(0)| &\le n\int_{-1/(2n)}^{1/(2n)} |\varphi(x)-\varphi(0)|\,dx \\ &\le n\cdot\frac{1}{n}\sup_{|x|\le 1/(2n)} |\varphi(x)-\varphi(0)| \\ &\xrightarrow{n\to\infty} 0.\end{aligned}$$

Weak Convergence of (f_n)

Hence

$$T_{f_n}\varphi \xrightarrow{n\to\infty} \varphi(0) = T_\delta\varphi$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$. Therefore,

$$T_{f_n} \xrightarrow{n\to\infty} T_\delta.$$

Formally,

$$f_n \xrightarrow{n\to\infty} \delta$$

in the sense of distributions.

Since (f_n) is a physical model for a point source, this shows:

A physical point source is represented by the Dirac distribution T_δ.

Criterion for Weak Convergence

Lemma. Suppose

- $f_n \in L^1_{\text{loc}}(\mathbb{R}^n)$
- For any $R > 0$, $\sup_{|x|<R} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

Then

$$f_n \to f \qquad \text{distributionally.}$$

Criterion for Weak Convergence

Proof.
Suppose $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\operatorname{supp}\varphi \subset \{x\colon |x| < R\}$ for some $R > 0$

$$\begin{aligned} |T_{f_n}\varphi - T_f\varphi| &\le \int_{\mathbb{R}^n} |f_n(x) - f(x)| \cdot |\varphi(x)|\, dx \\ &\le \underbrace{\sup_{|x|<R} |f_n(x) - f(x)|}_{\to 0} \cdot \underbrace{\int_{|x|<R} |\varphi(x)|\, dx}_{=:C} \end{aligned}$$

□

Note:
Pointwise convergence is not necessary and not sufficient for distributional comnvergence.