Two Applications of the Weak Derivative

$f(x) = 1/x$ as a Distribution

Problem:

$$f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}, \qquad f(x) = \frac{1}{x}$$

is not locally integrable.

But we would like to have a distribution analogous to this function!

Approach:

$$g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}, \qquad g(x) = \ln(|x|)$$

is locally integrable. Furthermore,

$$g'(x) = f(x) \qquad \text{and} \qquad T_g' \quad \text{exists.}$$

Distributional Derivative of the Logarithm

$$
\begin{aligned}
T_g'\varphi &= -\int_{-\infty}^{\infty} \varphi'(x)\ln(|x|)\,dx \\
&= -\int_{-\infty}^{0} \varphi'(x)\ln(-x)\,dx - \int_{0}^{\infty} \varphi'(x)\ln(x)\,dx \\
&= -\int_{0}^{\infty} \varphi'(-x)\ln(x)\,dx - \int_{0}^{\infty} \varphi'(x)\ln(x)\,dx \\
&= \lim_{\varepsilon\to 0}\left(-\int_{\varepsilon}^{\infty} (\varphi'(x)+\varphi'(-x))\ln(x)\,dx\right) \\
&= \lim_{\varepsilon\to 0}\left(-(\varphi(x)-\varphi(-x))\ln(x)\Big|_{\varepsilon}^{\infty} + \int_{\varepsilon}^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx\right) \\
&= \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx
\end{aligned}
$$

The Principle Value of $1/x$

Definition. The distribution $\mathcal{P}(1/x) \in \mathcal{D}'(\mathbb{R})$ is given by

$$\mathcal{P}\left(\frac{1}{x}\right)(\varphi) := \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\, dx$$

The right-hand side is called

- the Cauchy principal part integral of φ or
- the Cauchy principal value of $\displaystyle\int_{\mathbb{R}} \frac{\varphi(x)}{x}\, dx$.

The Cauchy principal part integral converges for any $\varphi \in \mathcal{D}(\mathbb{R})$.

The Laplacian of $1/|x|$ in $\mathbb{R}^3$

$$f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \qquad\qquad f(x) = \frac{1}{|x|}$$

is locally integrable (even about the origin - use polar coordinates!)

The Laplacian in $\mathbb{R}^n$ is the linear differential operator

$$\Delta := \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$$

Then

$$\begin{aligned}(\Delta T_f)(\varphi) = T_f(\Delta\varphi) &= \int_{\mathbb{R}^3} \frac{\Delta\varphi(x)}{|x|}\,dx \\ &= \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{\Delta\varphi(x)}{|x|}\,dx\end{aligned}$$

The Laplacian of $1/|x|$ in $\mathbb{R}^3$

Green's second identity:

$$\int_{|x|>\varepsilon} \frac{\Delta\varphi(x)}{|x|}\,dx = \int_{|x|>\varepsilon} \varphi(x)\,\Delta\left(\frac{1}{|x|}\right)\,dx$$
$$+\int_{|x|=\varepsilon}\left(\frac{1}{|x|}\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)\right)\,d\sigma$$

normal derivative (inward pointing)

Spherical coordinates (r, ϕ, θ):

$$\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$$

and

$$\Delta\left(\frac{1}{|x|}\right) = \Delta\frac{1}{r} = 0 \qquad\qquad \text{for } r \neq 0$$

The Laplacian of $1/|x|$ in $\mathbb{R}^3$

Hence

$$\int_{|x|>\varepsilon} \frac{\Delta\varphi(x)}{|x|}\,dx = -\int_{r=\varepsilon}\left(\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\varphi}{r^2}\right)\,d\sigma.$$

$\varphi \in \mathcal{D}(\mathbb{R}^3)$ implies $\frac{\partial\varphi}{\partial r}$ bounded, so

$$\left|\int_{r=\varepsilon}\frac{1}{r}\frac{\partial\varphi}{\partial r}\,d\sigma\right| \le \frac{\text{constant}}{\varepsilon}\cdot 4\pi\varepsilon^2 \xrightarrow{\varepsilon\to 0} 0$$

Using spherical coordinates,

$$\int_{r=\varepsilon}\frac{\varphi}{r^2}\,d\sigma \xrightarrow{\varepsilon\to 0} 4\pi\varphi(0).$$

The Laplacian of $1/|x|$ in $\mathbb{R}^3$

In summary:

$$(\Delta T_f)(\varphi) = \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\Delta\varphi(x)}{|x|}\, dx = -4\pi\varphi(0),$$

so that

$$\Delta \frac{1}{|x|} = -4\pi\delta(x)$$

in the sense of distributions.