Elementary Operations on Distributions

Extension by Duality

Basic Idea:

- Any operation in calculus can be performed on test functions $\varphi \in \mathcal{D} = C_0^\infty$.
- Use the "dual pairing"

$$T_g \varphi = \int_{\mathbb{R}^n} g(x)\varphi(x)\, dx$$

  to see what equivalent operation can be performed on "sufficiently nice" $g \in L^1_{\text{loc}}$.
- Define the operation on distributions in terms of an equivalent operation on test functions.

Dilation

Dilation operator: For $\alpha > 0$, define

$$D_\alpha \colon \mathcal{D}(\mathbb{R}^n) \mapsto \mathcal{D}(\mathbb{R}^n), \qquad (D_\alpha \varphi)(x) = \alpha^{n/2} \cdot \varphi(\alpha x).$$

Then for a regular distribution T_g,

$$\begin{aligned} T_g(D_\alpha \varphi) &= \alpha^{n/2} \int_{\mathbb{R}^n} g(x) \varphi(\alpha x)\, dx \\ &= \frac{1}{\alpha^{n/2}} \int_{\mathbb{R}^n} g\left(\frac{x}{\alpha}\right) \varphi(x)\, dx \\ &= T_{D_{1/\alpha} g} \varphi \\ &=: (D_{1/\alpha} T_g) \varphi. \end{aligned}$$

Dilation

Definition. The dilation operator

$$D_\alpha \colon \mathcal{D}'(\mathbb{R}^n) \mapsto \mathcal{D}'(\mathbb{R}^n)$$

is defined by

$$(D_\alpha T)(\varphi) := T(D_{1/\alpha}\varphi).$$

This definition ensures that

$$T_{D_\alpha g} = D_\alpha T_g$$

and extends the definition of the dilation from $L^1_{\text{loc}}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$.

Translation

Translation operator: For $y \in \mathbb{R}^n$, define

$$\tau_y \colon \mathcal{D}(\mathbb{R}^n) \mapsto \mathcal{D}(\mathbb{R}^n), \qquad (\tau_y \varphi)(x) = \varphi(x - y).$$

Then for a regular distribution T_g,

$$T_g(\tau_y \varphi) = \int_{\mathbb{R}^n} g(x)\varphi(x - y)\, dx = \int_{\mathbb{R}^n} g(x + y)\varphi(x)\, dx = T_{\tau_{-y} g}\varphi.$$

Definition. We define the translation operator

$$\tau_y \colon \mathcal{D}'(\mathbb{R}^n) \mapsto \mathcal{D}'(\mathbb{R}^n) \qquad (\tau_y T)(\varphi) := T(\tau_{-y}\varphi)$$

The Translation of Distributions

Example. Let $T_\delta \varphi = \varphi(0)$. Then

$$(\tau_\xi T_\delta)(\varphi) = T_\delta(\tau_{-\xi}\varphi) = \varphi(x+\xi)\big|_{x=0} = \varphi(\xi).$$

Sum and Scalar Multiplication

$\mathcal{D}'(\mathbb{R}^n)$ is a vector space:

- $T_1, T_2 \in \mathcal{D}'$ implies $T_1 + T_2 \in \mathcal{D}'$ with

$$(T_1 + T_2)(\varphi) = T_1\varphi + T_2\varphi$$

- $T \in \mathcal{D}'$, $\lambda \in \mathbb{C}$ implies $\lambda T \in \mathcal{D}'$ with

$$(\lambda T)(\varphi) = \lambda \cdot T\varphi$$

The pointwise sum and scalar multiple of functions generalize automatically to distributions:

$$T_{f+g} = T_f + T_g, \qquad\qquad T_{\lambda f} = \lambda T_f.$$

Multiplication by Smooth Functions

Multiplication operator: For $h \in C^\infty(\mathbb{R}^n)$, define

$$M_h \colon \mathcal{D}(\mathbb{R}^n) \mapsto \mathcal{D}(\mathbb{R}^n), \qquad (M_h\varphi)(x) = h(x)\varphi(x).$$

Then for a regular distribution T_g,

$$T_g(M_h\varphi) = \int_{\mathbb{R}^n} g(x)h(x)\varphi(x)\,dx = T_{M_h g}\varphi.$$

Definition. We define the mutliplication operator

$$M_h \colon \mathcal{D}'(\mathbb{R}^n) \mapsto \mathcal{D}'(\mathbb{R}^n) \qquad (M_h T)(\varphi) := T(M_h\varphi)$$

Warning: We can not multiply a distribution with a non-smooth function or another distribution!