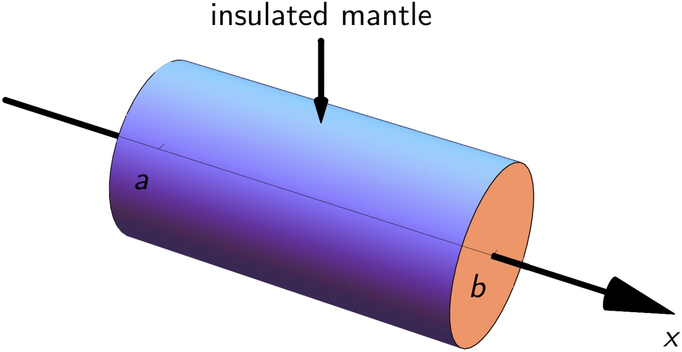
The Classical Heat Equation

The Heat Equation for a Cylinder

Consider a cylinder with an insulated mantle, as shown below:



We are interested in the temperature θ within the cylinder.

The Heat Equation for a Cylinder

Assumption: The temperature is a function of x and t only,

$$\theta = \theta(x,t), \qquad\qquad x \in [a,b], \quad t \geq 0.$$

Heat density H related to temperature via

$$H(x,t) = \underset{\substack{\nearrow \\ \text{material density}}}{\varrho} \cdot \underset{\substack{\nwarrow \\ \text{specific heat capacity}}}{c} \cdot \theta(x,t).$$

$$\text{Total heat} = \int_a^b H(x,t)\,dx.$$

Change of Heat in the Cylinder

Change in heat is due to two physical processes:

(i) Heat is generated from heat sources.

$$\text{Heat created at time } t = Q(t).$$

(ii) Heat enters or leaves through the faces of the cylinder.

Heat flux in the positive x-direction:

$$B(x,t) = -\underset{\substack{\nearrow \\ \text{heat conduction coefficient}}}{k}\frac{\partial\theta(x,t)}{\partial x}$$

(Fourier's law of heat conduction)

Change of Heat in the Cylinder

Total heat change between time t and time $t + \Delta t$:

$$\int_a^b H(x, t + \Delta t)\, dx - \int_a^b H(x, t)\, dx$$
$$= \int_t^{t+\Delta t} \big(B(a, \tau) - B(b, \tau) + Q(\tau)\big)\, d\tau$$

We divide the equation by Δt and let $\Delta t \to 0$ to obtain the instantaneous change in heat:

$$\int_a^b \frac{\partial H}{\partial t}\, dx = B(a, t) - B(b, t) + Q(t)$$

Change of Heat in the Cylinder

The heat-temperature relation and Fourier's law give

$$\begin{aligned}\varrho c \int_a^b \frac{\partial \theta}{\partial t}\,dx &= k\,\frac{\partial \theta}{\partial x}\bigg|_{x=b} - k\,\frac{\partial \theta}{\partial x}\bigg|_{x=a} + Q(t)\\ &= k\int_a^b \frac{\partial^2 \theta}{\partial x^2}\,dx + Q(t)\end{aligned}$$

(Fundamental equation for the temperature)

Assumption: The heat sources can be expressed in terms of an integrable heat source density q, i.e.,

$$Q(t) = \int_a^b q(x,t)\,dx.$$

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Then

$$\varrho c \int_a^b \frac{\partial \theta}{\partial t}\, dx = \int_a^b \left(k \frac{\partial^2 \theta}{\partial x^2} + q(x,t) \right) dx.$$

Letting $a = x$, $b = x + \varepsilon$, divide by ε and let $\varepsilon \to 0$,

$$\frac{\partial \theta}{\partial t} = \alpha^2 \frac{\partial^2 \theta}{\partial x^2} + \frac{1}{\varrho c} q(x,t),$$

with thermal diffusivity

$$\alpha^2 = \frac{k}{\varrho c} > 0.$$

(Classical heat equation)