Vv557 Methods of Applied Mathematics II

Green Functions and Boundary Value Problems

Assignment 14 (Selected Solutions)

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Exercise 14.1 Laplace Equation on the Infinite Strip
We consider a Dirichlet problem for the negative Laplace operator on the infinite strip $S = \mathbb{R} \times (0, a)$, $a > 0$, in $\mathbb{R}^2$, i.e., the problem

$$-\Delta u = 0, \qquad x \in S, \qquad u\big|_{x_2=0} = 0, \qquad u\big|_{x_2=a} = f(x_1), \qquad x_1 \in \mathbb{R}, \tag{1}$$

where we assume that f is a bounded function. We have obtained Green's function in terms of a formal series of image charges,

$$g(x;\xi) = \sum_{n\in\mathbb{Z}} \big(E(x;\xi_{2n}^+) - E(x;\xi_{2n}^-)\big) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \big(\ln\big(|x-\xi_{2n}^+|\big) - \ln\big(|x-\xi_{2n}^-|\big)\big)$$

where

$$\xi_{2n}^\pm = (\xi_1, 2an \pm \xi_2).$$

i) Show that the series in fact converges.

ii) Calculate the sum of the series to obtain

$$g(x;\xi) = \frac{1}{4\pi} \ln\left(1 + \frac{\cos\left(\frac{\pi}{a}(x_2+\xi_2)\right) - \cos\left(\frac{\pi}{a}(x_2-\xi_2)\right)}{\cosh\left(\frac{\pi}{a}(x_1-\xi_1)\right) - \cos\left(\frac{\pi}{a}(x_2+\xi_2)\right)}\right).$$

iii) Verify that

$$-\frac{\partial g(x;\xi)}{\partial \xi_2}\bigg|_{\xi_2=a} = \frac{1}{2a} \frac{\sin\left(\frac{\pi}{a}x_2\right)}{\cosh\left(\frac{\pi}{a}(x_1-\xi_1)\right) + \cos\left(\frac{\pi}{a}x_2\right)}$$

in agreement with the results obtained previously by the partial eigenfunction expansions.

Solution.

i) To verify the convergence of the series, note that

$$\begin{aligned}|x-\xi_{2n}^\pm|^2 &= (x_1-\xi_1)^2 + (x_2 \mp \xi_2 - 2an)^2 \\ &= (x_1-\xi_1)^2 + (x_2\mp\xi_2)^2 - 4an(x_2\mp\xi_2) + 4a^2n^2\end{aligned}$$

Then

$$\begin{aligned}|x-\xi_{2n}^\pm| &= \sqrt{(x_1-\xi_1)^2 + (x_2\mp\xi_2)^2 - 4an(x_2\mp\xi_2) + 4a^2n^2} \\ &= 2a|n|\sqrt{1 + \frac{(x_1-\xi_1)^2 + (x_2\mp\xi_2)^2}{4a^2n^2} - \frac{(x_2\mp\xi_2)}{an}}\end{aligned}$$

and

$$\ln\big(|x-\xi_{2n}^\pm|\big) = \ln(2a|n|) - \frac{x_2\mp\xi_2}{2an} + O\left(\frac{1}{n^2}\right) \qquad \text{as } n\to\infty.$$

It follows that

$$\begin{aligned}g(x;\xi) &= \frac{1}{2\pi} \lim_{N\to\infty} \sum_{n=-N}^{N} \big(\ln\big(|x-\xi_{2n}^+|\big) - \ln\big(|x-\xi_{2n}^-|\big)\big) \\ &= \frac{1}{2\pi}\ln\left(\frac{|x-\xi_0^+|}{|x-\xi_0^-|}\right) + \lim_{N\to\infty}\sum_{n=0}^{N}\big[\ln\big(|x-\xi_{2n}^+|\big) - \ln\big(|x-\xi_{2n}^-|\big) + \ln\big(|x-\xi_{-2n}^+|\big) - \ln\big(|x-\xi_{-2n}^-|\big)\big] \\ &= \frac{1}{2\pi}\ln\left(\frac{|x-\xi_0^+|}{|x-\xi_0^-|}\right) + \lim_{N\to\infty}\sum_{n=0}^{N}\left[-\frac{x_2-\xi_2}{2an} + \frac{x_2+\xi_2}{2an} + \frac{x_2-\xi_2}{2an} - \frac{x_2-\xi_2}{2an} + O\left(\frac{1}{n^2}\right)\right]\end{aligned}$$

and the series converges, because all terms that are proportional to $1/n$ cancel.

ii) At this point it becomes very convenient to introduce complex numbers.[1] We define

$$z := x_1 + ix_2, \qquad \zeta := \xi_1 + i\xi_2.$$

Then, purely algebraically, we have

$$|x - \xi_{2n}^+|^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2 - 2an)^2 = |z - \zeta - 2ian|^2,$$
$$|x - \xi_{2n}^-|^2 = (x_1 - \xi_1)^2 + (x_2 + \xi_2 - 2an)^2 = |z - \overline{\zeta} - 2ian|^2$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$ as usual and the absolute value on the left refers to that oif $\mathbb{R}^2$ while the one on the right refers to that of $\mathbb{C}$.

The convergence of the series and the continuity of the logarithm guarantee that we can write

$$\begin{aligned} g(x;\xi) &= \frac{1}{2\pi}\ln\left(\prod_{n\in\mathbb{Z}}\frac{|z-\zeta-2ian|}{|z-\overline{\zeta}-2ian|}\right) \\ &= \frac{1}{2\pi}\ln\left(\frac{\prod_{n\in\mathbb{Z}}|z-\zeta-2ian|}{\prod_{n\in\mathbb{Z}}|z-\overline{\zeta}-2ian|}\right) \\ &= \frac{1}{2\pi}\ln\left(\frac{|z-\zeta|\prod_{n\in\mathbb{Z}\setminus\{0\}}|z-\zeta-2ian|}{|z-\overline{\zeta}|\prod_{n\in\mathbb{Z}\setminus\{0\}}|z-\overline{\zeta}-2ian|}\right) \end{aligned}$$

The trick is now to use the Weierstrass factorization of the hyperbolic sine, given by

$$w\prod_{n\in\mathbb{Z}\setminus\{0\}}\left(1-\frac{w}{\pi in}\right) = \sinh(w) \qquad \text{for any } w \in \mathbb{C}.$$

So we put our expression in the correct form as follows:

$$\begin{aligned} g(x;\xi) &= \frac{1}{2\pi}\ln\left(\frac{|z-\zeta|\prod_{n\in\mathbb{Z}\setminus\{0\}}2a|n||\frac{z-\zeta}{2ian}-1|}{|z-\overline{\zeta}|\prod_{n\in\mathbb{Z}\setminus\{0\}}2a|n||\frac{z-\overline{\zeta}}{2ian}-1|}\right) \\ &= \frac{1}{2\pi}\ln\left(\frac{\frac{\pi}{2a}|z-\zeta|\prod_{n\in\mathbb{Z}\setminus\{0\}}|1-\frac{z-\zeta}{2ian}|}{\frac{\pi}{2a}|z-\overline{\zeta}|\prod_{n\in\mathbb{Z}\setminus\{0\}}|1-\frac{z-\overline{\zeta}}{2ian}|}\right) \\ &= \frac{1}{2\pi}\ln\left|\frac{w^+\prod_{n\in\mathbb{Z}\setminus\{0\}}(1-\frac{w^+}{\pi in})}{w^-\prod_{n\in\mathbb{Z}\setminus\{0\}}(1-\frac{w^-}{\pi in})}\right| \end{aligned}$$

with

$$w^+ := \frac{\pi}{2a}(z-\zeta), \qquad w^- := \frac{\pi}{2a}(z-\overline{\zeta}).$$

We then have

$$g(x;\xi) = \frac{1}{2\pi}\ln\left|\frac{\sinh(w^+)}{\sinh(w^-)}\right| = \frac{1}{4\pi}\ln\left(\frac{|\sinh\left(\frac{\pi}{2a}(z-\zeta)\right)|^2}{|\sinh\left(\frac{\pi}{2a}(z-\overline{\zeta})\right)|^2}\right)$$

Recall that

$$\cosh(w) := \frac{1}{2}(e^w + e^{-w}), \qquad \sinh(w) = \frac{1}{2}(e^w + e^{-w}), \qquad w \in \mathbb{C}.$$

From this we can deduce the analogous addition theorem to that of trigonomentric functions

$$\sinh(a+b) = \sinh(a)\cosh(b) + \sinh(b)\cosh(a), \qquad a, b \in \mathbb{C}$$

and also note that $\cosh(ix) = \cos(x)$, $\sinh(ix) = i\sin(x)$ for $x \in \mathbb{R}$. Everything together gives

$$\sinh(x+iy) = \sinh(x)\cos(y) + i\sin(y)\cosh(x)$$

[1] Credit for this part is to Cai Runze, TA for Vv557 in Spring 2018 and many years before that, for writing out the details.

so that, with $\sinh^2(x) = \cosh^2(x) - 1$,

$$\begin{aligned}|\sinh(x+iy)|^2 &= \sinh^2(x)\cos^2(y) + \sinh^2(y)\cosh^2(x) \\ &= \cosh^2(x)\cos^2(y) - \cos^2(y) + \sin^2(y)\cosh^2(x) \\ &= \cosh^2(x) - \cos^2(y).\end{aligned}$$

This implies that

$$\begin{aligned}\left|\sinh\left(\frac{\pi}{2a}(z-\zeta)\right)\right|^2 &= \cosh^2\left(\frac{\pi}{2a}(x_1-\xi_1)\right) - \cos^2\left(\frac{\pi}{2a}(x_2-\xi_2)\right), \\ \left|\sinh\left(\frac{\pi}{2a}(z-\overline{\zeta})\right)\right|^2 &= \cosh^2\left(\frac{\pi}{2a}(x_1-\xi_1)\right) - \cos^2\left(\frac{\pi}{2a}(x_2+\xi_2)\right),\end{aligned}$$

so

$$g(x;\xi) = \frac{1}{4\pi}\ln\left(\frac{\cosh^2\left(\frac{\pi}{2a}(x_1-\xi_1)\right) - \cos^2\left(\frac{\pi}{2a}(x_2-\xi_2)\right)}{\cosh^2\left(\frac{\pi}{2a}(x_1-\xi_1)\right) - \cos^2\left(\frac{\pi}{2a}(x_2+\xi_2)\right)}\right).$$

Since $\cos^2(w) = \frac{1}{2}(1+\cos(2w))$, $w \in \mathbb{C}$, we further simplify to

$$\begin{aligned}g(x;\xi) &= \frac{1}{4\pi}\ln\left(\frac{\cosh\left(\frac{\pi}{a}(x_1-\xi_1)\right) - \cos\left(\frac{\pi}{a}(x_2-\xi_2)\right)}{\cosh\left(\frac{\pi}{a}(x_1-\xi_1)\right) - \cos\left(\frac{\pi}{a}(x_2+\xi_2)\right)}\right) \\ &= \frac{1}{4\pi}\ln\left(1 + \frac{\cos\left(\frac{\pi}{a}(x_2+\xi_2)\right) - \cos\left(\frac{\pi}{a}(x_2-\xi_2)\right)}{\cosh\left(\frac{\pi}{a}(x_1-\xi_1)\right) - \cos\left(\frac{\pi}{a}(x_2+\xi_2)\right)}\right).\end{aligned}$$

Note that g vanishes when $x_2 = 0$ and when $x_2 = a$, as required.