Vv557 Methods of Applied Mathematics II

Green Functions and Boundary Value Problems

Assignment 13 (Selected Solutions)

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Exercise 13.1 Laplace Equation on the Infinite Strip

We consider a Dirichlet problem for the negative Laplace operator on the infinite strip $S = \mathbb{R} \times (0, a)$, $a > 0$, in $\mathbb{R}^2$, i.e., the problem

$$-\Delta u = 0, \qquad x \in S, \qquad u\big|_{x_2=0} = 0, \qquad u\big|_{x_2=a} = f(x_1), \qquad x_1 \in \mathbb{R}, \tag{1}$$

where we assume that f is a bounded function. We know from Section 12 that

$$u(x) = -\int_{-\infty}^{\infty} f(\xi_1)\, \frac{\partial g(x_1, x_2; \xi_1, \xi_2)}{\partial \xi_2}\bigg|_{\xi_2=a} d\xi_1. \tag{2}$$

i) Find both partial eigenfunction expansions of g.

ii) Simplify / sum the expansions to show that a bounded solution u is given by

$$u(x) = \frac{\sin(\pi x_2/a)}{2a} \int_{-\infty}^{\infty} \frac{f(\xi_1)}{\cos(\pi x_2/a) + \cosh((x_1 - \xi_1)\pi/a)}\, d\xi_1.$$

regardless of which eigenfunction expansion is used. You may use that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sinh(\alpha x_2)}{\sinh(\alpha a)} e^{i\alpha(x_1-\xi_1)}\, d\alpha = \frac{1}{2a} \frac{\sin(\pi x_2/a)}{\cos(\pi x_2/a) + \cosh((x_1 - \xi_1)\pi/a)}.$$

Solution. Green's function satisfies

$$\begin{aligned} -\Delta g(x;\xi) &= \delta(x - \xi), & x \in S, \\ g(x;\xi)\big|_{x_2=0} &= g(x;\xi)\big|_{x_2=a} = 0, & x_1 \in \mathbb{R}. \end{aligned} \tag{3}$$

for $\xi \in S$.

Partial Eigenfunction Expansion

We consider first the equation

$$-\Delta u = 0$$

and separate variables, setting $u(x_1, x_2) = X_1(x_1) \cdot X_2(x_2)$. Then

$$X_1'' X_2 + X_1 X_2'' = 0 \qquad \text{so} \qquad \frac{X_1''}{X_1} = -\frac{X_2''}{X_2} =: \lambda.$$

This gives

$$X_1'' = \lambda X_1, \qquad -\infty < x_1 < \infty, \tag{4}$$

$$X_2'' = -\lambda X_2, \qquad 0 < x_2 < a, \qquad X_2(0) = X_2(a) = 0. \tag{5}$$

For the first problem, there are no real boundary conditions. Let us assume, however, that the solution u should be bounded, so we require

$$\sup_{x\in\mathbb{R}} |X_1(x)| < \infty.$$

We easily find eigenfunctions and -values

$$X_{2,n}(x_2) = \sin(n\pi x_2/a), \qquad \lambda = \left(\frac{n\pi}{a}\right)^2, \qquad n \in \mathbb{N}$$

for (5). For (4) we note that solutions must be both bounded and differentiable, leading to

$$X_{1,\alpha}(x_1) = e^{i\alpha x_1}, \qquad \lambda = -\alpha^2 < 0, \qquad \alpha \in \mathbb{R}.$$

We do not relate the eigenvalues λ of these two problems with each other since we would like to find the partial eigenfunction expansions of g in terms of each problem separately, without regard to the other one.

Expansion in the X_2 Eigenfunctions

We expand g in terms of the X_2 eigenfunctions to obtain

$$g(x;\xi) = \sum_{n=1}^{\infty} g_n(x_1;\xi) \sin\left(\frac{n\pi x_2}{a}\right),$$

where

$$g_n(x_1;\xi) = \frac{2}{a}\int_0^a g(x;\xi)\sin\left(\frac{n\pi x_2}{a}\right) dx_2.$$

We now calculate the functions $g_n(x_1;\xi)$. We multiply (3) by $\frac{2}{a}\sin(n\pi x_2/a)$ and integrate from 0 to a with respect to x_2, obtaining

$$-g_n''(x_1;\xi) + \frac{n^2\pi^2}{a^2} g_n(x_1;\xi) = \frac{2}{a}\sin\left(\frac{n\pi\xi_2}{a}\right)\delta(x_1-\xi_1).$$

We now need to solve the ODE problem for a Green's function. One way to do this via a causal fundamental solution.

Consider the equation

$$-\frac{a}{2\sin\left(\frac{n\pi\xi_2}{a}\right)} E''(x_1) + \frac{n^2\pi^2}{2a\sin\left(\frac{n\pi\xi_2}{a}\right)} E(x_1) = \delta(x_1).$$

(We consider $\xi = 0$.) Then we first find u such that

$$u'' - \frac{n^2\pi^2}{a^2}u = 0, \qquad u(0) = 0, \qquad u'(0) = -\frac{2\sin\left(\frac{n\pi\xi_2}{a}\right)}{a}$$

Two independent solutions of the homogeneous equation are given by

$$u_1(x_1) = e^{n\pi x_1/a}, \qquad u_2(x) = e^{-n\pi x_1/a}$$

and we satisfy the initial conditions by taking

$$u(x_1) = \frac{\sin\left(\frac{n\pi\xi_2}{a}\right)}{n\pi}\left(e^{-n\pi x_1/a} - e^{n\pi x_1/a}\right)$$

and a causal fundamental solution is given by

$$E(x_1) = H(x_1)u(x_1)$$

where H is the Heaviside function. For our concrete problem, we need g (and hence g_n) to be bounded with respect to x_1, however, so we add a suitable solution to the homogeneous problem to eliminate the exponential increase as $x_1 \to \infty$,

$$u_{\text{hom}}(x_1) = \frac{\sin\left(\frac{n\pi\xi_2}{a}\right)}{n\pi} e^{n\pi x_1/a}$$

Then

$$g_n(x_1;\xi) = E(x_1-\xi_1) + u_{\text{hom}}(x_1-\xi_1) = \frac{e^{-(n\pi/a)|x_1-\xi_1|}}{n\pi}\sin\left(\frac{n\pi\xi_2}{a}\right)$$

We therefore have he representation

$$g(x;\xi) = \sum_{n=1}^{\infty}\frac{e^{-(n\pi/a)|x_1-\xi_1|}}{n\pi}\sin\left(\frac{n\pi x_2}{a}\right)\sin\left(\frac{n\pi\xi_2}{a}\right) \tag{6}$$

Note that

$$\left|\frac{e^{-(n\pi/a)|x_1-\xi_1|}}{n\pi}\sin\left(\frac{n\pi x_2}{a}\right)\sin\left(\frac{n\pi\xi_2}{a}\right)\right| \le e^{-(\pi/a)|x_1-\xi_1|}\cdot\frac{1}{n\pi}e^{-\pi(n-1)|x_1-\xi_1|/a}$$

so that for $x_1 \neq \xi_1$ the series converges uniformly by the Weierstraß M-test. Furthermore, g converges to zero as $\xi_1 \to \pm\infty$. An analogous estimate for the derivatives of the summands allows us to conclude that we can write

$$- \left.\frac{\partial g(x;\xi)}{\partial \xi_2}\right|_{\xi_2=a} = \frac{1}{a}\sum_{n=1}^{\infty}(-1)^n e^{-n\pi|x_1-\xi_1|/a}\sin\left(\frac{n\pi x_2}{a}\right) \tag{7}$$

and that the derivative also converges to zero as $\xi_1 \to \pm\infty$.

It follows that the solution formula (2) is valid and

$$u(x) = \frac{1}{a}\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{n\pi x_2}{a}\right)\int_{-\infty}^{\infty} f(\xi_1)e^{-n\pi|x_1-\xi_1|/a}\,d\xi_1 \tag{8}$$

The representation (8) is not very useful; for example, we can not easily verify that $u(x_1, a) = f(x_1)$ because every term in the series vanishes when $x_2 = a$. In particular, the series does not converge uniformly for $0 < x_2 < a$, so calculations will be cumbersome.

We can improve the behavior of our series solution by reconsidering the series in (7) as a geometric series:

$$\begin{aligned} - \left.\frac{\partial g(x;\xi)}{\partial \xi_2}\right|_{\xi_2=a} &= \frac{1}{a}\sum_{n=1}^{\infty}(-1)^n e^{-n\pi|x-\xi_1|/a}\sin\left(\frac{n\pi x_2}{a}\right) \\ &= -\frac{1}{a}\,\mathrm{Im}\Big(\sum_{n=1}^{\infty}\theta^n\Big) \end{aligned}$$

where

$$\theta = -e^{i\pi x_2/a}e^{-\pi|x-\xi_1|/a}$$

We then find

$$\begin{aligned} - \left.\frac{\partial g(x;\xi)}{\partial \xi_2}\right|_{\xi_2=a} &= -\frac{1}{a}\,\mathrm{Im}\,\frac{\theta}{1-\theta} = \frac{1}{a}\frac{\sin\left(\frac{\pi x_2}{a}\right)e^{-\pi|x-\xi_1|/a}}{1+2e^{-\pi|x-\xi_1|/a}\cos\left(\frac{\pi x_2}{a}\right)+e^{-2\pi|x-\xi_1|/a}} \\ &= \frac{1}{2a}\frac{\sin(\pi x_2/a)}{\cos(\pi x_2/a)+\cosh((x_1-\xi_1)\pi/a)} \end{aligned}$$

We then have the solution formula

$$u(x) = \frac{\sin(\pi x_2/a)}{2a}\int_{-\infty}^{\infty}\frac{f(\xi_1)}{\cos(\pi x_2/a)+\cosh((x_1-\xi_1)\pi/a)}\,d\xi_1. \tag{9}$$

Expansion in the X_1 Eigenfunctions

We have a continuous 1-parameter family of eigenfunctions

$$X_{1,\alpha}(x_1) = e^{i\alpha x_1}, \qquad \alpha \in \mathbb{R},$$

rather than a countable set. The expansion of Green's function must take the form of an integral instead of a series:

$$g(x,\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(\alpha, x_2;\xi)e^{i\alpha x_1}\,d\alpha \tag{10}$$

We now need to determine the "coefficient functions" $\hat{g}(\alpha, x_2;\xi)$.

We take the Fourier transform of (3) with respect to the x_1-variable and obtain

$$-\frac{\partial \hat{g}(\alpha, x_2;\xi)}{\partial x_2^2} + \alpha^2\hat{g}(\alpha, x_2;\xi) = \frac{e^{-i\alpha\xi_1}}{\sqrt{2\pi}}\delta(x_2-\xi_2) \tag{11}$$

where $\hat{g}(\alpha, x_2;\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\alpha x_1}g(x;\xi)\,dx_1$.[1] We find that

$$\hat{g}(\alpha, 0;\xi) = \hat{g}(\alpha, a;\xi) = 0$$

[1] We could also have obtained (11) by plugging (10) into (5).

since g vanishes for $x_2 = 0$ and $x_2 = a$. Finding $\hat{g}(\alpha, x_2; \xi)$ is hence a Green's function Dirichlet problem for the ODE (11). An easy calculation yields,

$$\hat{g}(\alpha, x_2; \xi) = \frac{e^{-i\alpha\xi_1}}{\sqrt{2\pi}\alpha \sinh(\alpha a)} \sinh(\alpha y_<) \sinh(\alpha(a - y_>)),$$

where

$$y_< := \min\{x_2, \xi_2\}, \qquad y_> := \max\{x_2, \xi_2\}.$$

It follows that

$$g(x; \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\alpha(x_1-\xi_1)}}{\sqrt{2\pi}\alpha \sinh(\alpha a)} \sinh(\alpha y_<) \sinh(\alpha(a - y_>))\, d\alpha \tag{12}$$

We now calculate

$$\begin{aligned} -\left.\frac{\partial g(x;\xi)}{\partial \xi_2}\right|_{\xi_2=a} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sinh(\alpha x_2)}{\sinh(\alpha a)} e^{i\alpha(x_1-\xi_1)}\, d\alpha \\ &= \frac{1}{2a} \frac{\sin(\pi x_2/a)}{\cos(\pi x_2/a) + \cosh((x_1 - \xi_1)\pi/a)} \end{aligned}$$

where the last identity can be shown using residue calculus (see the addendum below). This shows that the two eigenfunction expansions yield the same solution formula (of course, the Green's functions themselves can also be shown to be the same).

Addendum: a Complex Integral

We want to show that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sinh(\alpha x_2)}{\sinh(\alpha a)} e^{i\alpha(x_1-\xi_1)}\, d\alpha = \frac{1}{2a} \frac{\sin(\pi x_2/a)}{\cos(\pi x_2/a) + \cosh((x_1 - \xi_1)\pi/a)}.$$

For simplicity, we transform variables in the integrand, setting $y = \alpha a$. Then the integral on the left equals

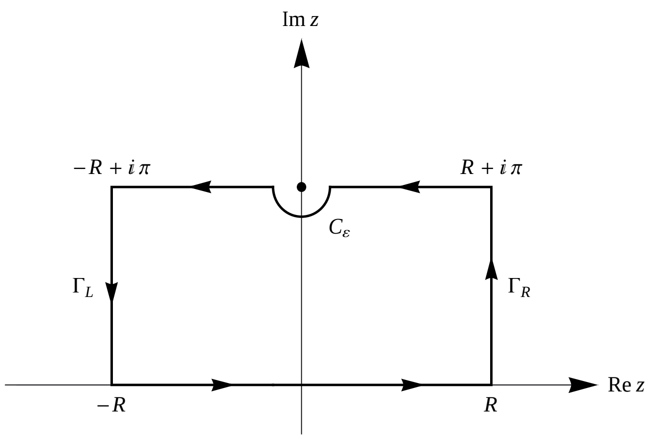
$$\frac{1}{a} \int_{-\infty}^{\infty} \frac{\sinh(y x_2/a)}{\sinh(y)} e^{iy(x_1-\xi_1)/a}\, dy$$

We plan to integrate the function

$$f(z) := \frac{\sinh(z x_2/a)}{\sinh(z)} e^{iz(x_1-\xi_1)/a}$$

over a suitable toy contour.

We note that the poles of $\sinh(zx_2/a)/\sinh(z)$ are at $z = in\pi$, $n \in \mathbb{Z} \setminus \{0\}$. (The point $z = 0$ is a removable singularity since $\lim_{z\to 0} \sinh(zx_2/a)/\sinh(z)$ exists.) We will use the following contour:



We parametrize Γ_R by $\gamma_R(t) = R + it$, $t \in [0, \pi]$. Then

$$\begin{aligned}
\left|\int_{\Gamma_R} f(z)\,dz\right| &\le \int_0^\pi \left|\frac{\sinh((R+it)x_2/a)}{\sinh(R+it)} e^{i(R+it)(x_1-\xi_1)/a}\right| dt \\
&\le \sup_{t\in[0,\pi]} \left|\frac{\sinh((R+it)x_2/a)}{\sinh(R+it)}\right| \int_0^\pi e^{-t(x_1-\xi_1)/a}\,dt \\
&\le C \cdot \sup_{t\in[0,\pi]} \left|\frac{e^{(R+it)x_2/a} - e^{-(R+it)x_2/a}}{e^{R+it} - e^{-(R+it)}}\right| \\
&= C \cdot e^{R(x_2/a-1)} \underbrace{\sup_{t\in[0,\pi]} \left|\frac{e^{itx_2/a} - e^{-(2R+it)x_2/a}}{e^{it} - e^{-(2R+it)}}\right|}_{\text{bounded}} \\
&\xrightarrow{R\to\infty} 0
\end{aligned}$$

since $x_2 < a$. The same calculation shows that the integral over Γ_L vanishes when $R \to \infty$.

Next, we consider the integral over the semi-circle C_ε. Since the semi-circle is traversed in clockwise direction,

$$\int_{C_\varepsilon} f(z)\,dz = -\pi i \operatorname{res}_{i\pi} f.$$

In the following calculations we will use that

$$i\sinh(it) = \sin t \qquad \text{and} \qquad \sinh(t + i\pi) = -\sinh(t)$$

In particular, we have

$$\begin{aligned}
\operatorname{res}_{i\pi} f &= \sinh(i\pi x_2/a)e^{-\pi(x_1-\xi_1)/a} \lim_{y\to i\pi} \frac{y - i\pi}{\sinh(y)} \\
&= \sinh(i\pi x_2/a)e^{-\pi(x_1-\xi_1)/a} \lim_{y\to 0} \frac{y}{\sinh(y + i\pi)} \\
&= \sinh(i\pi x_2/a)e^{-\pi(x_1-\xi_1)/a} \underbrace{\lim_{y\to 0} \frac{-y}{\sinh(y)}}_{=-1} \\
&= -\sinh(i\pi x_2/a)e^{-\pi(x_1-\xi_1)/a} \\
&= -i\sin(\pi x_2/a)e^{-\pi(x_1-\xi_1)/a}
\end{aligned}$$

Letting $\varepsilon \to 0$ and $R \to \infty$, we then see that

$$\int_{-\infty}^{\infty} \frac{\sinh(yx_2/a)}{\sinh(y)} e^{iy(x_1-\xi_1)/a}\,dy - \pi\sin(i\pi x_2/a)e^{-\pi(x_1-\xi_1)/a} - \int_{-\infty}^{\infty} \frac{\sinh((t+i\pi)x_2/a)}{\sinh(t+i\pi)} e^{i(t+i\pi)(x_1-\xi_1)/a}\,dt = 0,$$

where the last integral is over the line through $i\pi$ and parallel to the real axis. We note that

$$\begin{aligned}
\sinh((t+i\pi)x_2/a) &= i\sin((t/i+\pi)x_2/a) \\
&= i\sin((t/i)x_2/a)\cos(\pi x_2/a) + i\cos((t/i)x_2/a)\sin(\pi x_2/a) \\
&= \sinh(tx_2/a)\cos(\pi x_2/a) + i\cosh(tx_2/a)\sin(\pi x_2/a).
\end{aligned}$$

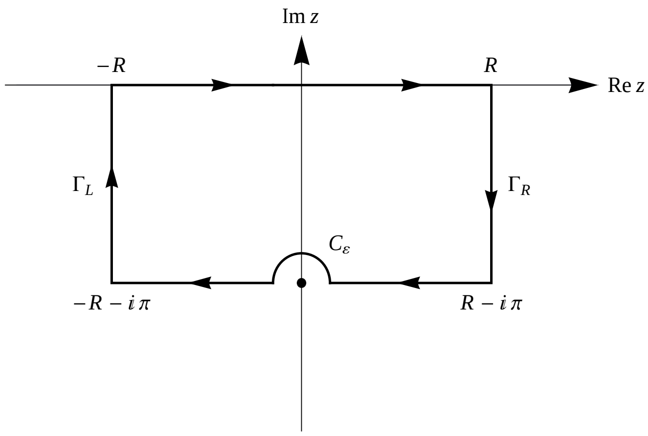
Then

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sinh(yx_2/a)}{\sinh(y)} e^{iy(x_1-\xi_1)/a}\,dy &= \pi\sin(\pi x_2/a)e^{-\pi(x_1-\xi_1)/a} \\
&\qquad - e^{-\pi(x_1-\xi_1)/a} \int_{-\infty}^{\infty} \frac{\sinh((t+i\pi)x_2/a)}{\sinh(t)} e^{it(x_1-\xi_1)/a}\,dt \\
&= \pi\sin(\pi x_2/a)e^{-\pi(x_1-\xi_1)/a} \\
&\qquad - e^{-\pi(x_1-\xi_1)/a}\cos(\pi x_2/a) \int_{-\infty}^{\infty} \frac{\sinh(tx_2/a)}{\sinh(t)} e^{it(x_1-\xi_1)/a}\,dt \\
&\qquad - ie^{-\pi(x_1-\xi_1)/a}\sin(\pi x_2/a) \int_{-\infty}^{\infty} \frac{\cosh(tx_2/a)}{\sinh(t)} e^{it(x_1-\xi_1)/a}\,dt
\end{aligned}$$

We hence have

$$\left(e^{\pi(x_1-\xi_1)/a}+\cos(\pi x_2/a)\right)\int_{-\infty}^{\infty}\frac{\sinh(yx_2/a)}{\sinh(y)}e^{iy(x_1-\xi_1)/a}\,dy$$
$$=\pi\sin(\pi x_2/a)-i\sin(\pi x_2/a)\int_{-\infty}^{\infty}\frac{\cosh(tx_2/a)}{\sinh(t)}e^{it(x_1-\xi_1)/a}\,dt \quad (*)$$

We now integrate the function $f(z)$ over the contour



As before, the integrals over Γ_L and Γ_R will vanish when $R\to\infty$ and the integral over C_ε around the pole at $-i\pi$ contributes one-half the residue. We note

$$\begin{aligned}\operatorname{res}_{-i\pi} f &= -\sinh(i\pi x_2/a)e^{\pi(x_1-\xi_1)/a}\lim_{y\to -i\pi}\frac{y+i\pi}{\sinh(y)}\\ &= -\sinh(i\pi x_2/a)e^{\pi(x_1-\xi_1)/a}\lim_{y\to 0}\frac{y}{\sinh(y-i\pi)}\\ &= -\sinh(i\pi x_2/a)e^{\pi(x_1-\xi_1)/a}\underbrace{\lim_{y\to 0}\frac{-y}{\sinh(y)}}_{=-1}\\ &= \sinh(i\pi x_2/a)e^{\pi(x_1-\xi_1)/a}\\ &= i\sin(i\pi x_2/a)e^{\pi(x_1-\xi_1)/a}\end{aligned}$$

Finally, since

$$\begin{aligned}\sinh((t-i\pi)x_2/a) &= i\sin((t/i-\pi)x_2/a)\\ &= i\sin((t/i)x_2/a)\cos(\pi x_2/a)-i\cos((t/i)x_2/a)\sin(\pi x_2/a)\\ &= \sinh(tx_2/a)\cos(\pi x_2/a)-i\cosh(tx_2/a)\sin(\pi x_2/a)\end{aligned}$$

we find

$$\left(e^{\pi(x_1-\xi_1)/a}+\cos(\pi x_2/a)\right)\int_{-\infty}^{\infty}\frac{\sinh(yx_2/a)}{\sinh(y)}e^{iy(x_1-\xi_1)/a}\,dy$$
$$=\pi\sin(\pi x_2/a)+i\sin(\pi x_2/a)\int_{-\infty}^{\infty}\frac{\cosh(tx_2/a)}{\sinh(t)}e^{it(x_1-\xi_1)/a}\,dt \quad (**)$$

Adding $(*)$ and $(**)$, we find

$$\int_{-\infty}^{\infty}\frac{\sinh(yx_2/a)}{\sinh(y)}e^{iy(x_1-\xi_1)/a}\,dy=\frac{2\pi\sin(\pi x_2/a)}{e^{-\pi(x_1-\xi_1)/a}+e^{\pi(x_1-\xi_1)/a}+2\cos(\pi x_2/a)}$$

After dividing by $2\pi a$, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sinh(\alpha x_2)}{\sinh(\alpha a)}e^{i\alpha(x_1-\xi_1)}\,d\alpha=\frac{1}{2a}\frac{\sin(\pi x_2/a)}{\cos(\pi x_2/a)+\cosh((x_1-\xi_1)\pi/a)}.$$