Vv557 Methods of Applied Mathematics II

Green Functions and Boundary Value Problems

Assignment 6

Exercise 6.1 Simply Supported Beam

We want to find a fundamental solution of the stationary equation for a simply supported beam, i.e., a function $g(x, \xi)$ satisfying

$$\frac{d^4 g}{dx^4} = \delta(x - \xi), \qquad 0 < x, \xi < 1,$$

with boundary conditions

$$g(0, \xi) = g''(0, \xi) = g(1, \xi) = g''(1, \xi) = 0.$$

i) Find a causal fundamental solution, i.e., a function E satisfying

$$\frac{d^4 E}{dx^4} = \delta(x - \xi), \qquad 0 < x, \xi < 1,$$

and $E(x) = 0$ for $x < \xi$.

ii) Add a solution of the homogeneous equation $\frac{d^4 u}{dx^4} = 0$ to E to obtain a function that satisfies the boundary conditions.

Exercise 6.2 Equilibrium Diffusion

The equilibrium concentration u of a substance diffusing in a homogeneous, absorbing, infinite, one-dimensional medium (such as an infinite tube) is given by

$$Lu = -\frac{d^2 u}{dx^2} + q^2 u = f(x), \qquad x \in \mathbb{R},$$

where f is the source density of the substance and $q > 0$ is a positive constant.

i) Let $\xi \in \mathbb{R}$ be fixed. Use the Fourier transform to find a fundamental solution $E(x; \xi)$ of L satisfying

$$LE(x; \xi) = \delta(x - \xi), \qquad \lim_{|x| \to \infty} E(x, \xi) = 0. \tag{1}$$

Is this a causal fundamental solution? Why or why not?

ii) Verify that the candidate function found satisfies (1) distributionally.

Exercise 6.3 Traveling Wave

The goal of this exercise is to obtain a fundamental solution of the stationary equation for a traveling wave with wavenumber k, i.e., a function $g(x, \xi)$ satisfying

$$-\frac{d^2 g}{dx^2} - k^2 g = \delta(x - \xi), \qquad 0 < x, \xi < 1,$$

with boundary conditions

$$g(0, \xi) = g(1, \xi) = 0.$$

i) Find a causal fundamental solution, i.e., a function E satisfying

$$-\frac{d^2 E}{dx^2} - k^2 E = \delta(x - \xi), \qquad 0 < x, \xi < 1,$$

and $E(x) = 0$ for $x < \xi$.

ii) Add a solution of the homogeneous equation $-\frac{d^2u}{dx^2} - k^2u = 0$ to E to obtain a function that satisfies the boundary conditions.

iii) Use the Fourier transform to find a fundamental solution on $\mathbb{R}$, i.e., a function E satisfying

$$-\frac{d^2E}{dx^2} - k^2E = \delta(x-\xi), \qquad x, \xi \in \mathbb{R}.$$

iv) Add a solution of the homogeneous equation $-\frac{d^2u}{dx^2} - k^2u = 0$ to E to obtain a function that satisfies the boundary conditions.