Vv557 Methods of Applied Mathematics II

Green Functions and Boundary Value Problems

Assignment 5

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Exercise 5.3
Consider the wave equation problem for a function $u\colon \mathbb{R}^2 \to \mathbb{R}$,

$$u_{tt} - u_{xx} = 0, \qquad u(x,0) = f(x), \qquad u_t(x,0) = g(x).$$

Take the Fourier transform of the equation with respect to the x-variable to obtain an ODE in the t-variable and solve the ODE to obtain

$$\widehat{u}(\xi,t) = \hat{f}(\xi)\cos(\xi t) + \frac{\hat{g}(\xi)}{\xi}\sin(\xi t).$$

Then calculate the inverse Fourier transform (in the distributional sense) to obtain a solution formula for $u(x,t)$.

Solution. Taking the Fourier transform, we obtain

$$\widehat{u}_{tt} + \xi^2\widehat{u} = 0, \qquad \widehat{u}(\xi,0) = \widehat{f}(\xi), \qquad \widehat{u}_t(\xi,0) = \widehat{g}(\xi).$$

The ODE has solution

$$\widehat{u}(\xi,t) = c_1\cos(\xi t) + c_2\sin(\xi t), \qquad c_1, c_2 \in \mathbb{R}.$$

Then the initial conditions give

$$\widehat{u}(\xi,0) = c_1 = \widehat{f}(\xi), \qquad \widehat{u}_t(\xi,0) = c_2\xi = \widehat{g}(\xi),$$

so

$$\begin{aligned}
\widehat{u}(\xi,t) &= \hat{f}(\xi)\cos(\xi t) + \frac{\hat{g}(\xi)}{\xi}\sin(\xi t) \\
&= \frac{1}{2}e^{i\xi t}\hat{f}(\xi) + \frac{1}{2}e^{-i\xi t}\hat{f}(\xi) + \frac{\hat{g}(\xi)}{\xi}\sin(\xi t).
\end{aligned}$$

We now use that $e^{i\xi t}\hat{f}(\xi) = \mathcal{F}(f(\,\cdot\, - t))$ to see that

$$u(x,t) = \frac{f(x-t)+f(x+t)}{2} + t\mathcal{F}^{-1}_{\xi\to x}\left(\frac{\sin(\xi t)}{\xi t}\hat{g}(\xi)\right).$$

Using $t\mathcal{F}(f(t\cdot(\,\cdot\,))) = (\mathcal{F}f)(\,\cdot\,/t)$ and

$$\widehat{\Pi}(\xi) = \frac{2}{\sqrt{2\pi}}\sin(\xi)/\xi, \qquad \Pi(x) = \begin{cases} 1 & |x| < 1, \\ 0 & \text{otherwise} \end{cases}$$

we see that

$$\begin{aligned}
u(x,t) &= \frac{f(x-t)+f(x+t)}{2} + t\mathcal{F}^{-1}_{\xi\to x}\left(\frac{\sin(\xi t)}{\xi t}\hat{g}(\xi)\right) \\
&= \frac{f(x-t)+f(x+t)}{2} + \frac{\sqrt{2\pi}}{2}\mathcal{F}^{-1}_{\xi\to x}\left(\widehat{\Pi}(\xi/t)\hat{g}(\xi)\right) \\
&= \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}(\Pi(\,\cdot\,/t) * g)(x) \\
&= \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}\int_{-\infty}^{\infty}\Pi(y/t)g(x-y)\,dy \\
&= \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}\int_{-t}^{t}g(x-y)\,dy \\
&= \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t}g(y)\,dy
\end{aligned}$$