Vv557 Methods of Applied Mathematics II

Green Functions and Boundary Value Problems

Assignment 2 (Selected Solutions)

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Exercise 2.1
Let $\xi \in (0,1)$ be fixed. The goal of this exercise is to show that the Green's function $g(x,\xi)$ (introduced and defined in the lecture video) for the problem

$$-u'' = f(x), \qquad 0 < x < 1, \qquad\qquad u(0) = u(1) = 0$$

satisfies

$$-g'' = \delta(x-\xi), \qquad 0 < x < 1, \tag{1}$$

in the distributional sense. This will require the definition of distributions on the open set $\Omega = (0,1) \subset \mathbb{R}$.

Proceed as follows: Define first $\mathcal{D}(0,1) := \{\varphi \in \mathcal{D}(\mathbb{R})\colon \operatorname{supp}\varphi \subset (0,1)\}$ and then $\mathcal{D}'(0,1)$ as the set of continuous linear functionals on $\mathcal{D}(0,1)$. Regard $g(\,\cdot\,,\xi)$ as an element of $\mathcal{D}'(0,1)$ and differentiate it as a distribution. Note that the test functions will have compact support in the interval $(0,1) \subset \mathbb{R}$.

Solution. Recall that the Green's function $g(x,\xi)$ is given by

$$g(x,\xi) = \begin{cases} (1-\xi)x, & 0 \le x < \xi, \\ (1-x)\xi, & \xi \le x \le 1. \end{cases}$$

For any $\varphi \in \mathcal{D}(0,1)$, since $\operatorname{supp}\varphi \subset (0,1)$, φ must vanish on the boundary, i.e., $\varphi(0) = \varphi(1) = 0$. By the definition of distributional derivative

$$\begin{aligned}
-T_g''\varphi = -T_g\varphi'' &= -\int_0^1 g(x,\xi)\varphi''s(x)dx \\
&= -\int_0^\xi (1-\xi)x\varphi''(x)dx - \int_\xi^1 (1-x)\xi\varphi''(x)dx \\
&= -(1-\xi)\left[x\varphi'(x)\Big|_0^\xi - \int_0^\xi \varphi'(x)dx\right] - \xi\left[(1-x)\varphi'(x)\Big|_\xi^1 - \int_\xi^1 \varphi'(x)d(1-x)\right] \\
&= -(1-\xi)[\xi\varphi'(\xi) - 0 - (\varphi(\xi)-\varphi(0))] - \xi[0 - (1-\xi)\varphi'(\xi) + (\varphi(1)-\varphi(\xi))] \\
&= \varphi(\xi) \\
&= T_{\delta_\xi}\varphi
\end{aligned}$$

hence $-g'' = \delta(x-\xi)$ in the distributional sense.

Exercise 2.2

i) Verify that the Cauchy principal value $\mathcal{P}(1/x)$ defines a distribution, i.e., that it is a continuous linear functional on $\mathcal{D}(\mathbb{R})$.

ii) Verify that $x\mathcal{P}(1/x) = 1$ in the sense of distributions.

Exercise 2.3
Show that

$$\frac{d}{dx}\mathcal{P}\left(\frac{1}{x}\right) = -\mathcal{P}\left(\frac{1}{x^2}\right),$$

where $\mathcal{P}(1/x^2) \in \mathcal{D}'(\mathbb{R})$ is defined by

$$\mathcal{P}\left(\frac{1}{x^2}\right)(\varphi) := \lim_{\varepsilon \searrow 0}\int_{|x|>\varepsilon} \frac{\varphi(x)-\varphi(0)}{x^2}\,dx.$$

Solution. The principal value integral for $1/x$ acting on $\varphi \in \mathcal{D}(\mathbb{R})$ is given by

$$\mathcal{P}\left(\frac{1}{x}\right)(\varphi) = \lim_{\varepsilon \searrow 0} \int_{|x|>\varepsilon} \frac{1}{x}\varphi(x)\,dx$$

By definition,

$$-\mathcal{P}\left(\frac{1}{x}\right)'(\varphi) = \mathcal{P}\left(\frac{1}{x}\right)(\varphi') = \lim_{\varepsilon \searrow 0} \int_{|x|>\varepsilon} \frac{1}{x}\varphi'(x)\,dx$$

For any $\varepsilon > 0$,

$$\begin{aligned}
\int_{|x|>\varepsilon} \frac{1}{x}\varphi'(x)\,dx &= \int_{\varepsilon}^{\infty} \frac{1}{x}\varphi'(x)\,dx + \int_{-\infty}^{\varepsilon} \frac{1}{x}\varphi'(x)\,dx \\
&= \frac{\varphi(x)}{x}\Big|_{\varepsilon}^{\infty} + \int_{\varepsilon}^{\infty} \frac{1}{x^2}\varphi(x)\,dx + \frac{\varphi(x)}{x}\Big|_{-\infty}^{-\varepsilon} + \int_{-\infty}^{-\varepsilon} \frac{1}{x^2}\varphi(x)\,dx \\
&= -\frac{\varphi(\varepsilon)}{\varepsilon} - \frac{\varphi(-\varepsilon)}{\varepsilon} + \int_{|x|>\varepsilon} \frac{1}{x^2}\varphi(x)\,dx \\
&= -\frac{\varphi(\varepsilon)-\varphi(0)}{\varepsilon} + \frac{\varphi(-\varepsilon)-\varphi(0)}{-\varepsilon} - 2\frac{\varphi(0)}{\varepsilon} + \int_{|x|>\varepsilon} \frac{1}{x^2}\varphi(x)\,dx \\
&= -\frac{\varphi(\varepsilon)-\varphi(0)}{\varepsilon} + \frac{\varphi(-\varepsilon)-\varphi(0)}{-\varepsilon} + \int_{|x|>\varepsilon} \frac{1}{x^2}(\varphi(x)-\varphi(0)\,dx
\end{aligned}$$

Here we have used

$$\frac{2}{\varepsilon} = \int_{-\infty}^{-\varepsilon} \frac{1}{x^2}\,dx + \int_{\varepsilon}^{\infty} \frac{1}{x^2}\,dx = \int_{|x|>\varepsilon} \frac{1}{x^2}\,dx.$$

Note that

$$\lim_{\varepsilon \searrow 0} \frac{\varphi(\varepsilon)-\varphi(0)}{\varepsilon - 0} = \varphi'(0) = \lim_{\varepsilon \searrow 0} \frac{\varphi(-\varepsilon)-\varphi(0)}{0-\varepsilon}$$

so

$$\begin{aligned}
-\mathcal{P}\left(\frac{1}{x}\right)'(\varphi) &= -\lim_{\varepsilon \searrow 0} \frac{\varphi(\varepsilon)-\varphi(0)}{\varepsilon} + \lim_{\varepsilon \searrow 0} \frac{\varphi(-\varepsilon)-\varphi(0)}{-\varepsilon} + \lim_{\varepsilon \searrow 0} \int_{|x|>\varepsilon} \frac{1}{x^2}(\varphi(x)-\varphi(0))\,dx \\
&= \lim_{\varepsilon \searrow 0} \int_{|x|>\varepsilon} \frac{1}{x^2}(\varphi(x)-\varphi(0))\,dx = \mathcal{P}\left(\frac{1}{x^2}\right)(\varphi)
\end{aligned}$$

Exercise 2.4
Show that

$$g \in \mathcal{D}'(\mathbb{R}^2), \qquad\qquad g(x) = -\frac{1}{2\pi}\log|x|$$

satisfies $-\Delta g = \delta(x)$ in the distributional sense.

Solution. We calculate the derivative in the sense of distributions and introduce a paaremeter $R > 0$ to treat the integral,

$$\begin{aligned}
(\Delta T_g)(\varphi) = T_g(\Delta\varphi) &= \int_{\mathbb{R}^2} g(x)\Delta\varphi(x)\,dx \\
&= \lim_{R\to 0} \int_{|x|\geq R} g(x)\Delta\varphi(x)\,dx.
\end{aligned}$$

By Green's second identity,

$$\int_{|x|\geq R} g(x)\Delta\varphi(x)\,dx = \int_{|x|\geq R} \Delta g(x)\varphi(x)\,dx + \int_{|x|=R} g(x)\frac{\partial\varphi}{\partial n}\,d\sigma - \int_{|x|=R} \varphi(x)\frac{\partial g}{\partial n}\,d\sigma$$

Let us introduce polar coordinates and write $\widetilde{g}(r,\theta)$ for $g(r\cos(\theta), r\sin(\theta))$. The Laplace operator in polar coordinates is given by

$$\Delta_{(r,\theta)} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

which simply means that

$$\Delta_{(r,\theta)} u(r\cos\theta, r\sin\theta) = \Delta u(x_1, x_2)\big|_{(x_1,x_2)=(r\cos\theta, r\sin\theta)}$$

if u is a twice-differentiable function on $\mathbb{R}^2$. Since $\widetilde{g}(r,\theta)$ depends only on r and in fact $\widetilde{g}'(r) = -1/(2\pi r)$, we have

$$\Delta_{(r,\theta)} g = \frac{1}{r}\frac{\partial}{\partial r}(rg'(r)) = 0$$

so the first integral on the right vanishes and

$$\int_{|x|\geq R} g(x)\Delta\varphi(x)\,dx = \int_{|x|=R} g(x)\frac{\partial\varphi}{\partial n}\,d\sigma - \int_{|x|=R} \varphi(x)\frac{\partial g}{\partial n}\,d\sigma$$

Furthermore,

$$\begin{aligned}\left|\int_{|x|=R} g(x)\frac{\partial\varphi}{\partial n}\,d\sigma\right| &\leq 2\pi R\cdot \sup_{|x|=R}\left|g(x)\frac{\partial\varphi}{\partial n}\right| \\ &\leq R\cdot\ln(R)\sup_{|x|=R}\left|\frac{\partial\varphi}{\partial R}\right| \\ &\xrightarrow{R\to 0} 0.\end{aligned}$$

and

$$\begin{aligned}\int_{|x|=R}\varphi(x)\frac{\partial g}{\partial n}\,d\sigma &= \int_{|x|=R}\varphi(x)\left(-\frac{\partial}{\partial r}\left(-\frac{1}{2\pi}\ln(r)\right)\right)\,d\sigma \\ &= \frac{1}{2\pi R}\int_{|x|=R}\varphi(x)\,d\sigma \\ &\xrightarrow{R\to 0}\varphi(0).\end{aligned}$$

so

$$\lim_{R\to 0}\int_{|x|\geq R} g(x)(-\Delta)\varphi(x)\,dx = \varphi(0).$$