Vv557 Methods of Applied Mathematics II

Green Functions and Boundary Value Problems

Assignment 1

Recall that a sequence (f_n) of functions $f_n \colon I \to \mathbb{C}$, where $I \subset \mathbb{R}$, converges *pointwise* to a function $f \colon I \to \mathbb{C}$ if

$$\lim_{n\to\infty} |f_n(x) - f(x)| = 0 \qquad \text{for all } x \in I.$$

The convergence is *uniform* if

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0.$$

Exercise 1.1
Let $\xi \in (0,1) \subset \mathbb{R}$ be fixed. Solve the problem

$$-u'' = f_n(x;\xi), \qquad 0 < x < 1, \qquad\qquad u(0) = u(1) = 0 \tag{1}$$

for

$$f_n(x;\xi) = \begin{cases} n, & |x-\xi| < 1/2n, \\ 0 & \text{otherwise,} \end{cases}$$

with $1/n$ smaller than $\min\{\xi, 1-\xi\}$.

i) For n as above, find the solution u_n of (1). You should obtain

$$u_n(x) = \begin{cases} (1-\xi)\cdot x & 0 \le x \le \xi - \frac{1}{2n}, \\ (1-\xi)\cdot x - \frac{n}{2}\big(x - \xi + 1/(2n)\big)^2 & \xi - \frac{1}{2n} < x < \xi + \frac{1}{2n}, \\ \xi \cdot (1-x) & \xi + \frac{1}{2n} \le x \le 1. \end{cases}$$

ii) Verify that the sequence of solutions $u_n(x;\xi)$ converges pointwise on $[0,1]$ as $n \to \infty$ to the Green function $g(x,\xi)$ derived in the lecture.

iii) Is the convergence uniform on $[0,1]$? Prove your assertion!

Exercise 1.2
Which of the following are distributions? Justify your response!

i) $T \colon \mathcal{D}(\mathbb{R}) \to \mathbb{C}$, $\varphi \mapsto \varphi(-10)$,

ii) $T \colon \mathcal{D}(\mathbb{R}) \to \mathbb{C}$, $\varphi \mapsto \varphi(0)^2$,

iii) $T \colon \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}^n$, $\varphi \mapsto \operatorname{grad}\varphi(0)$,

iv) $T \colon \mathcal{D}(\mathbb{R}) \to \mathbb{C}$, $\varphi \mapsto \varphi(0) + \varphi(1) + \varphi(2) + \varphi(3) + \dots$,

v) $T \colon \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$, $\varphi \mapsto \int_{S^{n-1}} \varphi$, where $S^{n-1} = \{x \in \mathbb{R}^n \colon |x| = 1\}$.

vi) $T_f \colon \mathcal{D}(\mathbb{R}) \to \mathbb{R}$, $\varphi \mapsto \int_{\mathbb{R}} f(x)\varphi(x)\,dx$, with

   (a) $f(x) = 1/x$,
   (b) $f(x) = 1/\sqrt{|x|}$,
   (c) $f(x) = 1/x^2$.