Vv557 Methods of Applied Mathematics II

Green Functions and Boundary Value Problems

Assignment 6 (Selected Solutions)

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Exercise 6.2 Equilibrium Diffusion

The equilibrium concentration u of a substance diffusing in a homogeneous, absorbing, infinite, one-dimensional medium (such as an infinite tube) is given by

$$Lu = -\frac{d^2u}{dx^2} + q^2 u = f(x), \qquad x \in \mathbb{R},$$

where f is the source density of the substance and $q > 0$ is a positive constant.

i) Let $\xi \in \mathbb{R}$ be fixed. Use the Fourier transform to find a fundamental solution $E(x;\xi)$ of L satisfying

$$LE(x;\xi) = \delta(x-\xi), \qquad \lim_{|x|\to\infty} E(x,\xi) = 0. \tag{1}$$

Is this a causal fundamental solution? Why or why not?

ii) Verify that the candidate function found satisfies (1) distributionally.

Solution.

i) Since the differential equation has constant coefficients, we can assume $\xi = 0$. Taking the Fourier transform of the equation

$$-\frac{d^2u}{dx^2} + q^2 u = \delta(x)$$

we obtain

$$|k|^2 \hat{u}(k) + q^2 \hat{u}(k) = \frac{1}{\sqrt{2\pi}}.$$

or

$$\hat{u}(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{k^2+q^2}.$$

The inverse Fourier transform is

$$u(x) = \frac{1}{2q} e^{-q|x|}$$

and the solution is

$$E(x;\xi) = \frac{1}{2q} e^{-q|x-\xi|}$$

The solution is not causal, since it does not vanish for $x < \xi$.

ii) We apply the left-hand side to a test function $\varphi \in \mathcal{D}(\mathbb{R})$:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \big(-E(x;\xi)\varphi''(x) + q^2 E(x;\xi)\varphi(x)\big)\, dx \\
&= \frac{1}{2q}\int_{-\infty}^{\infty} \big(-e^{-q|x-\xi|}\varphi''(x) + q^2 e^{-q|x-\xi|}\varphi(x)\big)\, dx \\
&= \frac{1}{2q}\int_{-\infty}^{\infty} \big(-e^{-q|x-\xi|}\varphi''(x) + q^2 e^{-q|x-\xi|}\varphi(x)\big)\, dx \\
&= -\frac{1}{2q}\int_{\xi}^{\infty} e^{-q(x-\xi)}\varphi''(x)\, dx - \frac{1}{2q}\int_{-\infty}^{\xi} e^{q(x-\xi)}\varphi''(x)\, dx + \frac{1}{2q}\int_{-\infty}^{\infty} q^2 e^{-q|x-\xi|}\varphi(x)\, dx \\
&= \underbrace{-\frac{1}{2q}e^{-q(x-\xi)}\varphi'(x)\big|_{\xi}^{\infty}}_{=\varphi'(\xi)/(2q)} - \frac{1}{2}\int_{\xi}^{\infty} e^{-q(x-\xi)}\varphi'(x)\, dx \underbrace{-\frac{1}{2q}e^{q(x-\xi)}\varphi'(x)\big|_{-\infty}^{\xi}}_{=-\varphi'(\xi)/(2q)} + \frac{1}{2}\int_{-\infty}^{\xi} e^{q(x-\xi)}\varphi'(x)\, dx \\
&\quad + \frac{q}{2}\int_{-\infty}^{\infty} e^{-q|x-\xi|}\varphi(x)\, dx \\
&= \underbrace{-\frac{1}{2}e^{-q(x-\xi)}\varphi(x)\big|_{\xi}^{\infty}}_{=\varphi(\xi)/2} - \frac{q}{2}\int_{\xi}^{\infty} e^{-q(x-\xi)}\varphi(x)\, dx + \underbrace{\frac{1}{2}e^{q(x-\xi)}\varphi(x)\big|_{-\infty}^{\xi}}_{=\varphi(\xi)/2} - \frac{q}{2}\int_{-\infty}^{\xi} e^{q(x-\xi)}\varphi(x)\, dx \\
&\quad + \frac{q}{2}\int_{-\infty}^{\infty} e^{-q|x-\xi|}\varphi(x)\, dx \\
&= \varphi(\xi) = \int_{\mathbb{R}} \delta(x-\xi)\varphi(x)\, dx.
\end{aligned}$$

Exercise 6.3 Traveling Wave

The goal of this exercise is to obtain a fundamental solution of the stationary equation for a traveling wave with wavenumber k, i.e., a function $g(x,\xi)$ satisfying

$$-\frac{d^2 g}{dx^2} - k^2 g = \delta(x-\xi), \qquad 0 < x, \xi < 1,$$

with boundary conditions

$$g(0,\xi) = g(1,\xi) = 0.$$

i) Find a causal fundamental solution, i.e., a function E satisfying

$$-\frac{d^2 E}{dx^2} - k^2 E = \delta(x-\xi), \qquad 0 < x, \xi < 1,$$

and $E(x) = 0$ for $x < \xi$.

ii) Add a solution of the homogeneous equation $-\frac{d^2 u}{dx^2} - k^2 u = 0$ to E to obtain a function that satisfies the boundary conditions.

iii) Use the Fourier transform to find a fundamental solution on $\mathbb{R}$, i.e., a function E satisfying

$$-\frac{d^2 E}{dx^2} - k^2 E = \delta(x-\xi), \qquad x, \xi \in \mathbb{R}.$$

iv) Add a solution of the homogeneous equation $-\frac{d^2 u}{dx^2} - k^2 u = 0$ to E to obtain a function that satisfies the boundary conditions.

Solution.

i) To find the a causal fundamental solution to the ODE,

$$-\frac{d^2 E}{dx^2} - k^2 E = \delta(x-\xi), \qquad 0 < x, \xi < 1,$$

we impose the initial condition $E(0;\xi) = E'(0;\xi) = 0$, then via Laplace transform,

$$-s^2\mathcal{E} - k^2\mathcal{E} = e^{-\xi s}$$

hence

$$\mathcal{E}(s) = -\frac{e^{-\xi s}}{s^2 + k^2}$$

then by inverse Laplace transform, we have the causal fundamental solution as

$$E(x;\xi) = -\frac{1}{k} H(x-\xi)\sin(k(x-\xi)) \tag{2}$$

ii) We know that the general solution to the ODE $-u'' - k^2 u = 0$ is $u = c_1 \cos kx + c_2 \sin kx$, with constants $c_1, c_2 \in \mathbb{R}$. Suppose that

$$g(x,\xi) = E(x;\xi) + c_1 \cos kx + c_2 \sin kx, \tag{3}$$

then by imposing the boundary conditions, we have

$$\begin{aligned} g(0,\xi) &= c_1 \cdot 1 + c_2 \cdot 0 = 0, \\ g(1,\xi) &= -\frac{1}{k}\sin(k(1-\xi)) + c_1 \cos k + c_2 \sin k, \end{aligned}$$

thus we have $c_1 = 0$ and $c_2 = \dfrac{\sin(k(1-\xi))}{k \sin k}$, hence the Green's function is given by

$$\begin{aligned} g(x,\xi) &= -\frac{1}{k}H(x-\xi)\sin(k(x-\xi)) + \frac{\sin(k(1-\xi))}{k\sin k}\sin kx \\ &= \begin{cases} \dfrac{\sin(k(1-\xi))}{k\sin k}\sin kx, & x < \xi \\ -\dfrac{1}{k}\sin(k(x-\xi)) + \dfrac{\sin(k(1-\xi))}{k\sin k}\sin kx, & x > \xi \end{cases} \end{aligned}$$

that is, after simplification,

$$g(x,\xi) = \begin{cases} \dfrac{\sin(k(1-\xi))\sin(kx)}{k\sin k}, & x < \xi \\ \dfrac{\sin(k(1-x))\sin(k\xi)}{k\sin k}, & x > \xi \end{cases} \tag{4}$$

iii) Given the ODE in Eq. (**??**), we apply Fourier transform and get an algebraic equation of $\hat{E}(\omega)$,

$$\omega^2 \hat{E} - k^2 \hat{E} = \frac{e^{-i\xi\omega}}{\sqrt{2\pi}}$$

thus

$$\hat{E}(\omega) = \frac{1}{\sqrt{2\pi}}\frac{e^{-i\xi\omega}}{\omega^2 - k^2} = \frac{e^{-i\xi\omega}}{2k\sqrt{2\pi}}\left[\frac{1}{\omega-k} - \frac{1}{\omega+k}\right] \tag{5}$$

Recall from the lecture the Fourier transform of the Heaviside function,

$$(\mathcal{F}H)(\omega) = \frac{i}{\sqrt{2\pi}}\mathcal{P}\left(\frac{1}{\omega}\right) + \sqrt{\frac{\pi}{2}}\delta(\omega),$$

since $\operatorname{sgn} x = 2H(x) - 1$, by linearity of Fourier transform, we have

$$(\mathcal{F}\operatorname{sgn})(\omega) = 2\left[\frac{i}{\sqrt{2\pi}}\mathcal{P}\left(\frac{1}{\omega}\right) + \sqrt{\frac{\pi}{2}}\delta(\omega)\right] - \sqrt{2\pi}\delta(\omega) = i\sqrt{\frac{2}{\pi}}\mathcal{P}\left(\frac{1}{\omega}\right)$$

Therefore by taking the inverse Fourier transform of (5), we have

$$\begin{aligned} E(x;\xi) &= \frac{i}{4k}e^{ik(x-\xi)}\operatorname{sgn}(x-\xi) + \frac{i}{4k}e^{-ik(x-\xi)}\operatorname{sgn}(-(x-\xi)) \\ &= -\frac{1}{2k}\sin(k|x-\xi|) \end{aligned}$$

iv) Again assume that $g(x,\xi)$ has the form in (3), and note that $0 < x, \xi < 1$, the boundary conditions gives,

$$\begin{aligned} g(0,\xi) &= -\frac{1}{2k}\sin(k\xi) + c_1 \cdot 1 + c_2 \cdot 0 = 0, \\ g(1,\xi) &= -\frac{1}{2k}\sin(k(1-\xi)) + c_1 \cos k + c_2 \sin k = 0, \end{aligned}$$

thus $c_1 = \frac{1}{2}\sin(k\xi)$ and $c_2 = \frac{\sin(k(1-\xi)) - \sin(k\xi)\cos k}{k\sin k}$, and thus

$$g(x,\xi) = -\frac{1}{2k}\sin(k|x-\xi|) + \left[\frac{1}{2}\sin(k\xi)\cos(kx) + \frac{\sin(k(1-\xi)) - \sin(k\xi)\cos k}{k\sin k}\right]\sin(kx),$$

which is the same as (4) after simplification.